Math 4100/6100 **Fall 2016**

Exam 2

Math 4100 students: Answer any THREE of the following FIVE questions
Math 6100 students: Answer any FOUR of the following FIVE questions

1. (a) Carefully state the *Mean Value Theorem* and use it to prove the following:
   i. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) = 0$ for all $x \in \mathbb{R}$, then f must be constant on $\mathbb{R}$.
   ii. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) \geq 0$ for all $x \in (0, \infty)$, then f is increasing on $(0, \infty)$.

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ has the property that
   $$|f(x) - f(y)| \leq |x - y|^2$$
   for all $x, y \in \mathbb{R}$. Prove that f is constant on $\mathbb{R}$.

   (c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, $f(0) = 0$, and f' is increasing on $(0, \infty)$. Prove that the function $g : (0, \infty) \to \mathbb{R}$ defined by
   $$g(x) = \frac{f(x)}{x}$$
   is increasing.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded.

   (a) Recall that the *oscillation of f at x* is define to be
   $$\operatorname{Osc}(f, x) := \lim_{\delta \to 0+} \sup_{y,z \in V_\delta(x)} |f(y) - f(z)|.$$
   Briefly explain why this is a well defined notion.

   (b) Prove that f is continuous at x if and only if $\operatorname{Osc}(f, x) = 0$.

   (c) Prove that for every $\varepsilon > 0$ the set $A_\varepsilon = \{x \in \mathbb{R} : \operatorname{Osc}(f, x) \geq \varepsilon\}$ is closed and deduce from this that the set of all points at which f is discontinuous is an F_σ set[1].

3. (a) Let $\{f_n\}$ is a sequence of continuous functions on a compact set $K \subseteq \mathbb{R}$. Prove that if $\{f_n\}$ is decreasing[2] and converges pointwise to 0 on K, then $f_n \to 0$ uniformly on K.

   (b) Prove that if $f_n \to f$ uniform on $[a, b]$ and each f_n is integrable, then f is also integrable and
   $$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$$

4. Let $\{r_n\}$ be any enumeration of all the rationals in $[0, 1]$ and define $f : [0, 1] \to \mathbb{R}$ by setting
   $$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = r_n \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases} .$$

   (a) Prove that $\lim_{x \to c} f(x) = 0$ for every $c \in [0, 1]$ and conclude that set of all points at which f is discontinuous is precisely $[0, 1] \cap \mathbb{Q}$.

   (b) Prove, directly from the definition, that f is integrable on $[0, 1]$ and $\int_0^1 f(x)\, dx = 0$.

5. (a) Prove that if f and g are two continuous functions on $[a, b]$ and $g(x) \geq 0$ for all $x \in [a, b]$, then there must exist a point $c \in (a, b)$ such that
   $$\int_a^b f(x)g(x)\, dx = f(c) \cdot \int_a^b g(x)\, dx.$$

   (b) Let f be a differentiable function on $[a, b]$.
   i. Prove that if $f'(a) < L < f'(b)$, then there must exist a point $c \in (a, b)$ such that $f'(c) = L$.
   ii. Conclude from part (i) above that f' cannot have any simple discontinuities on $[a, b]$.

[1] Recall that an F_σ set is a countable union of closed sets.
[2] by which we mean that $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$ and $n \in \mathbb{N}$.