Math 3100 **Spring 2018**

Additional Final Exam Practice Questions

1. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} a_n = L$.
   (a) Give the definition of $\lim_{n\to\infty} a_n = L$.
   (b) Use the definition of convergence given above to prove that
   $$\lim_{n\to\infty} \frac{3n+5}{n-3} = 3.$$
   (c) Give a direct proof, using the definition given in (a), of the fact that $\lim_{n\to\infty} a_n^2 = L^2$.
   (d) Prove that if $a_n < 10$ for all $n \in \mathbb{N}$, then $L \leq 10$ and give an example showing that for certain sequences $\{a_n\}$ the limit L could in fact equal 10.

2. (a) Carefully state the definition of the *supremum* (the least upper bound) of a set of real numbers and the *Axiom of Completeness* (the least upper bound axiom).
   (b) Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(0) = 0$ and $f(1) = 12$ and let
   $$A := \{x \in [0,1] \,:\, f(x) < 10\}.$$
   i. Prove that $\alpha := \sup(A)$ exists.
   ii. Show that there exists a sequence $\{\alpha_n\}$ in A with the property that $\lim_{n\to\infty} \alpha_n = \alpha$.
   iii. Conclude that $f(\alpha) \leq 10$.

   ** Bonus points: Show that in fact $f(\alpha) = 10$.

3. (a) Carefully state the *Monotone Convergence Theorem*.
   (b) Let $\{a_n\}$ be defined recursively by $a_1 = 1$ and $a_{n+1} = \dfrac{3a_n + 2}{a_n + 2}$ for each $n \in \mathbb{N}$.
   Prove that $\{a_n\}$ converges and find its limit.
   (c) Let $\{x_n\}$ be a bounded sequence of real numbers.
   i. Carefully state the definition of $\limsup_{n\to\infty} x_n$ and justify why it always exists for such sequences.
   ii. Prove that if $\{z_n\}$ is a sequence of real numbers such that $0 \leq z_n \leq x_n$ for all $n \in \mathbb{N}$, then
   $$\limsup_{n\to\infty} z_n \leq \limsup_{n\to\infty} x_n.$$

4. (a) Carefully state the definition of a sequence of real numbers $\{a_n\}$ being a Cauchy sequence.
   (b) Prove that every convergent sequence is a Cauchy sequence.
   (c) i. Prove, using the definition given in (a), that Cauchy sequences are always bounded.

ii. Carefully state the *Bolzano-Weierstrass Theorem* and use this to show that Cauchy sequences of real numbers are always convergent.

(d) i. State the so-called *Cauchy Criterion for Infinite Series.*

ii. Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it must also converge and satisfy

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|.$$

Hint: Show that for any $\varepsilon > 0$ *there exists an* N *such that* $\left| \sum_{n=N+1}^{\infty} a_n \right| \leq \sum_{n=N+1}^{\infty} |a_n| \leq \varepsilon.$

5. (a) State the ε-δ definition of $\lim_{x \to x_0} f(x) = L$.

(b) Determine the following limit and use the definition from part (a) to prove your answer:

$$\lim_{x \to 2} \frac{2x+1}{x^2+1}.$$

(c) Prove that $\lim_{x \to x_0} f(x) = L$ if and only if $\lim_{n \to \infty} f(x_n) = L$ for all sequences $\{x_n\}$ in $\mathbb{R} \setminus \{x_0\}$ with $\lim_{n \to \infty} x_n = x_0$.

(d) Let

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

i. Carefully argue that g is discontinuous at 0 (it is of course discontinuous at every point of $\mathbb{R}$).

ii. Let $h(x) = xg(x)$ for every $x \in \mathbb{R}$. Prove that h is continuous at 0, but is not differentiable at 0.

6. (a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0, then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0-h)}{2h}.$$

(b) i. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is three times differentiable on $[x_0, x_0+h]$, then

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(c)}{6}h^3$$

for some $c \in (x_0, x_0+h)$.

Hint: Apply the Generalized MVT to $f(x_0+h) - f(x_0) - f'(x_0)h - \frac{f''(x_0)}{2}h^2$ *&* h^3.

ii. Prove that if $f : \mathbb{R} \to \mathbb{R}$ has the property that f''' is continuous on (x_0-h, x_0+h), then

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{f'''(c)}{6}h^2$$

for some $c \in (x_0-h, x_0+h)$.