Infinite Series

1. Important infinite series

Geometric series: $\sum_{n=0}^{\infty} r^n$ converges $\iff$ $|r| < 1$. If $|r| < 1$, then $\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r}$.

The p-series: $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges $\iff$ $p > 1$.

2. Series Tests

Definition. Given a sequence $\{a_n\}$ let $s_n = a_1 + \cdots + a_n$ denote its nth partial sum, then

$$\{a_n\} \text{ summable} \iff \sum_{n=1}^{\infty} a_n \text{ converges} \iff \underline{\{s_n\} \text{ converges}}.$$

Theorem 1 (Cauchy Criterion)**.**

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \text{for every } \varepsilon > 0, \text{ there exists } N \text{ such that } \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon \text{ if } n > m > N.$$

Corollary 2. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem 3. *If $a_n \geq 0$ and $s_n = a_1 + \cdots + a_n$, then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \{s_n\} \text{ bounded.}$$

Theorem 4 (Cauchy Condensation Test)**.** *If $\{a_n\}$ is a decreasing sequence of non-negative terms, then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots \text{ converges.}$$

Theorem 5 (Direct Comparison Test)**.** *If $|a_n| \leq b_n$ for all sufficiently large $n \in \mathbb{N}$, then*

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Corollary 6 (Direct Comparison Test for Divergence)**.**

If $0 \leq b_n \leq a_n$ for all sufficiently large $n \in \mathbb{N}$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Corollary 7 (Absolute Convergence implies Convergence)**.**

If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Corollary 8 (Limit Comparison Test)**.** Suppose $a_n > 0$, $b_n > 0$, and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges.}$$

Theorem 9 (Root Test – Mainly a Theoretical Tool)**.** *Let* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

- *If* $\alpha < 1$, *then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*
- *If* $\alpha > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Recall, by considering for example $\sum \frac{1}{n}$ *and* $\sum \frac{1}{n^2}$, *that the Root Test is inconclusive if* $\alpha = 1$.

Theorem 10 (Ratio Test – A Computational Tool)**.** *Let* $\{a_n\}$ *be a sequence of non-zero terms.*

- *If* $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, *so in particular if* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, *then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*
- *If* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *infinitely often, so in particular if* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Recall, again by considering $\sum \frac{1}{n}$ *and* $\sum \frac{1}{n^2}$, *that the Ratio Test is inconclusive if* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Although Theorem 10 can easily be established directly, it can also deduced as a consequence of Theorem 9 via the following

Lemma 11. *If* $\{c_n\}$ *is any sequence of positive real numbers, then*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Finally we have:

Theorem 12 (Alternating Series Test)**.**

If $\{b_n\}$ *is decreasing with limit 0, then* $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$ *converges.*

3. Strategy for analyzing $\sum_{n=1}^{\infty} a_n$

1. Does $a_n \to 0$?

   If NO, then $\sum_{n=1}^{\infty} a_n$ <u>diverges</u>.

2. Does $\sum_{n=1}^{\infty} |a_n|$ converge?

   If YES, then $\sum_{n=1}^{\infty} a_n$ <u>converges absolutely</u>, and hence <u>converges</u>. Try using

   - geometric series and p-series
   - "direct" or "limit" comparison tests
   - ratio (or root) test
   - Cauchy condensation test (or integral test if you are familiar with that)

3. If $\sum_{n=1}^{\infty} |a_n|$ does not converge or you cannot decide, then try

   - alternating series test

   If this test applies, then $\sum_{n=1}^{\infty} a_n$ <u>converges</u>.

   Recall that if

   $\sum_{n=1}^{\infty} a_n$ <u>converges</u> but $\sum_{n=1}^{\infty} |a_n|$ <u>diverges</u>, then we say $\sum_{n=1}^{\infty} a_n$ <u>converges conditionally</u>.