

Sample Exam 3 – Version 2

No calculators. Show your work. Give full explanations. Good luck!

1. (7 points)

- (a) Carefully state the *Intermediate Value Theorem*.
- (b) Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$.
Prove that f must have a fixed point; that is, show that $f(x) = x$ for at least one value of $x \in [0, 1]$.

2. (15 points)

- (a) Carefully state the *Mean Value Theorem* and use it to prove the following:
 - i. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable with $f'(x) = 0$ for all $x \in \mathbb{R}$, then f must be constant on \mathbb{R} .
 - ii. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable with $f'(x) \geq 0$ for all $x \in (0, \infty)$, then f is increasing on $(0, \infty)$.
- (b) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ has the property that

$$|f(x) - f(y)| \leq |x - y|^2$$

for all $x, y \in \mathbb{R}$. Prove that f is constant on \mathbb{R} .

- (c) Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, $f(0) = 0$, and f' is increasing on $(0, \infty)$. Prove that the function $g : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$g(x) = \frac{f(x)}{x}$$

is increasing.

3. (10 points) Let $f(x) = \begin{cases} x^4 \sin(x^{-2}), & x \neq 0 \\ 0, & x = 0 \end{cases}$

- (a) Show that f is differentiable at 0 and compute $f'(x)$ for all $x \in \mathbb{R}$.
- (b) Is f' continuous at 0? Give your reasoning.
- (c) Is f' differentiable at 0? Give your reasoning.

4. (8 points)

- (a) Find the 4th order Maclaurin polynomial for $f(x) = \frac{\cos(x^2)}{1+x}$.
- (b) Use part (a) to find the value of $f^{(4)}(0)$ without differentiating.

5. (10 points)

- (a) Carefully state the *Lagrangian Remainder Estimate* for Maclaurin series.
- (b) Use the *Lagrangian Remainder Estimate* to determine the following:
 - i. An estimate for the accuracy of approximating $\sin x$ by $x - x^3/6$ when $|x| \leq 1/2$.
 - ii. Values of x for which the accuracy of approximating $\sin x$ by $x - x^3/6$ is less than 10^{-3} .

Note that you are not permitted to use the Alternating Series Remainder Estimate above.

- (c) Obtain, by any means, an estimate for the accuracy of approximating

$$\int_0^1 \frac{\sin x}{x} dx \quad \text{by} \quad 1 - \frac{1}{18}.$$

Math 3100 - Sample Exam 3 (Version 2) - SOLUTIONS

1. (a) Intermediate Value Theorem

Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous. If L is a real number between $f(a)$ & $f(b)$, then $\exists c \in (a, b)$ with $f(c) = L$.

(b) Claim

If f is a continuous function on $[0, 1]$ with range contained in $[0, 1]$, then $\exists x \in [0, 1]$ with $f(x) = x$.

Proof

Consider $g(x) = f(x) - x$ which is also continuous on $[0, 1]$.

Since $g(0) = f(0) - 0 = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 1 - 1 = 0$

IVT $\Rightarrow g(x) = 0$ for some $x \in [0, 1]$
 \uparrow
 $f(x) = x$.

□

2. (a) Mean Value Theorem

If f is carts on $[a, b]$ & diff'ble on (a, b) , then
 $\exists c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b-a)$.
 $\quad (*)$

Both (i) & (ii) follow immediately from $(*)$ which is valid,
for any pair $a, b \in \mathbb{R}$ & $(0, \infty)$ respectively for some c
between a & b .

(b) It follows that $\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y| \quad \forall x \neq y \text{ in } \mathbb{R}$

$$\Rightarrow \lim_{x \rightarrow y} \frac{f(x) - f(y)}{x - y} = 0 \quad \forall y \in \mathbb{R}$$

Squeeze Thm $\Rightarrow f'(y)$ exists and equals 0 for all $y \in \mathbb{R}$

1(a)(i) $\Rightarrow f$ constant.

(c) Since $g(x) = \frac{f(x)}{x} \Rightarrow g'(x) = \frac{x f'(x) - f(x)}{x^2} \quad \forall x \in (0, \infty)$.

It suffices, by 1(a)(ii), to show $f(x) \leq x f'(x) \quad \forall x \in (0, \infty)$.

But MVT $\Rightarrow \frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0} = f'(c)$ for some $c \in (0, x)$

The result follows as the fact that f' is increasing ensures that $f'(c) \leq f'(x)$.

3. Let $f(x) = \begin{cases} x^4 \sin(x^{-2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) • If $x \neq 0$, then $f'(x) = 4x^3 \sin(x^{-2}) - 2x \cos(x^{-2})$.

• Claim f is diff'ble at 0 with $f'(0) = 0$.

Proof $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} x^3 \sin(x^{-2}) = 0$

By Squeeze Thm
 (since $|x^3 \sin(x^{-2})| \leq |x|^3 \rightarrow 0$) \square

Thus $f'(x) = \begin{cases} 4x^3 \sin(x^{-2}) - 2x \cos(x^{-2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

(b) Claim f' is continuous at $x_0 = 0$.

Proof

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} [4x^3 \sin(x^{-2}) - 2x \cos(x^{-2})] \stackrel{\text{as } x \neq 0}{=} 0 = f'(0)$$

Again by Squeeze Theorem since

$$|4x^3 \sin(x^{-2}) - 2x \cos(x^{-2})| \leq 4|x|^3 + 2|x| \rightarrow 0. \quad \square$$

(c) Claim f' is not differentiable at $x_0 = 0$.

Proof

$$\lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f'(x)}{x} = \lim_{x \rightarrow 0} (4x^2 \sin(x^{-2}) - 2 \cos(x^{-2}))$$

as $x \neq 0$

Does not exist \square

Why?

This limit does not exist since $\lim_{x \rightarrow 0} 4x^2 \sin(x^{-2}) = 0$

by the squeeze theorem ($|4x^2 \sin(x^{-2})| \leq 4|x|^2 \rightarrow 0$) but

$\lim_{x \rightarrow 0} 2 \cos(x^{-2})$ does not exist (a fact that one can see

readily by considering the sequence $x_n = \frac{1}{\sqrt{2\pi n}}$ & $y_n = \frac{1}{\sqrt{\frac{\pi}{2} + 2\pi n}}$).

4. (a)

$$\begin{aligned}
 f(x) = \frac{\cos(x^2)}{1+x} &= \underbrace{(1-x+x^2-x^3+x^4-\dots)}_{\text{MacLaurin Series for } \frac{1}{1+x}} \underbrace{(1-\frac{x^4}{2}+\frac{x^8}{24}-\dots)}_{\text{MacLaurin series for } \cos(x^2)} \\
 &= 1-x+x^2-x^3+\left(\frac{1}{2}\right)x^4+\dots \\
 &\quad \swarrow \quad \searrow \\
 &\quad \text{"4th order MacLaurin Poly for } f \text{"}
 \end{aligned}$$

$$(b) f^{(4)}(0) = 4! \left(\frac{1}{2}\right) = \underline{\underline{12}}$$

5(a) Lagrangian Remainder Estimate for MacLaurin Series

If f is $(n+1)$ -times differentiable on $(-R, R)$, then for any $x \in (-R, R) \setminus \{0\}$ $\exists c$ between 0 & x such that

$$f(x) - \left[f(0) + f'(0)x + \dots + \frac{f^{(n)}(0)}{n!}x^n \right] = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

$$(b) (i) \left| \sin x - \left(x - \frac{x^3}{6}\right) \right| \leq \frac{|x|^5}{5!} \leq \frac{1}{5!} \left(\frac{1}{2}\right)^5 \text{ if } |x| \leq \frac{1}{2}.$$

↑
since $x - x^3/6$ is the 4th order MacLaurin poly for $\sin x$
(& $|\sin x| \leq 1$)

$$(ii) \left| \sin x - \left(x - \frac{x^3}{6}\right) \right| \leq \frac{|x|^5}{5!} \leq \frac{1}{1000} \text{ if } |x| \leq \sqrt[5]{\frac{5!}{1000}}$$

$$(c) \text{ Since } \sin x - \left(x - \frac{x^3}{6}\right) = \frac{\cos(c)}{120} x^5 \text{ for some } 0 < c < x$$

$$\Rightarrow \frac{\sin x}{x} - \left(1 - \frac{x^2}{6}\right) = \frac{\cos(c)}{120} x^4 \text{ for some } 0 < c < x$$

$$\Rightarrow \int_0^1 \frac{\sin x}{x} dx - \underbrace{\int_0^1 \left(1 - \frac{x^2}{6}\right) dx}_{= 1 - \frac{1}{6}} = \int_0^1 \frac{\cos(c)}{120} x^4 dx \leq \frac{1}{120} \int_0^1 x^4 dx = \underline{\underline{\frac{1}{600}}}$$