

## Sample Exam 3 – Version 1

*No calculators. Show your work. Give full explanations. Good luck!*

1. (4 points) Explain why there exist no examples of the following:

- (a) A continuous function on  $[0, 1]$  with range equal to  $(0, 1)$ .
- (b) A continuous function on  $[0, 1]$  with range equal to  $[0, 1] \cap \mathbb{Q}$

2. (8 points) Prove that  $\sum_{n=0}^{\infty} a_n x^n$  and  $\sum_{n=1}^{\infty} n a_n x^{n-1}$  have the same radius of convergence.

3. (8 points) Evaluate the following infinite series

$$(a) \sum_{n=1}^{\infty} \frac{n}{4^n} \quad (b) \sum_{n=1}^{\infty} \frac{(-1)^n}{n 4^n}$$

4. (15 points)

- (a) i. Find the sixth order Maclaurin polynomial for the function

$$f(x) = \frac{x^2}{2+x^2}$$

- ii. Without differentiating find the value of  $f^{(6)}(0)$ .

- (b) Let  $P_3(x)$  denote the third order Taylor polynomial centered at  $x_0 = 1$  of  $f(x) = \log x$ .

- i. Find  $P_3(x)$ .
- ii. Give an estimate for how well  $P_3(1.5)$  approximates  $\log(1.5)$ .

- (c) i. Carefully state the *Lagrangian Remainder Estimate* for Maclaurin series.

- ii. Find a polynomial that approximates  $e^x$  to within  $10^{-3}$  for all  $|x| \leq 1/2$ .

5. (15 points)

- (a) Carefully state what it means to say that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at  $x_0$  and prove that if  $f$  is differentiable at  $x_0$ , then  $f$  is continuous at  $x_0$ .

(b) Let  $h(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ .

- i. Prove that  $h$  is discontinuous at all  $x \neq 0$ .

- ii. Prove that  $h$  is differentiable at  $x = 0$ .

- iii. What can you say about the continuity of  $h$  at  $x = 0$  and the differentiability of  $h$  at  $x \neq 0$ ?

- (c) Let  $f : [a, b] \rightarrow \mathbb{R}$ .

Prove that if  $f$  has a minimum at a point  $c \in (a, b)$ , and if  $f'(c)$  exists, then  $f'(c) = 0$ .

6. (Bonus points) Let  $f : [a, b] \rightarrow \mathbb{R}$  be differentiable on  $[a, b]$ .

Prove that if  $f'(a) < 0 < f'(b)$ , then there exists  $c \in (a, b)$  such that  $f'(c) = 0$ .

Math 3100 - Sample Exam 3 (Version 1) - SOLUTIONS

1. (a) The Extreme Value Theorem implies that there cannot exist a continuous function on  $[0,1]$  with range equal to  $(0,1)$  since any continuous function on  $[a,b]$  must attain both a maximum and minimum value &  $(0,1)$  does not contain a max or min element.
- (b) The Intermediate Value Theorem implies that there cannot exist a continuous function on  $[0,1]$  with range equal to  $\mathbb{Q} \cap [0,1]$  since between any two rationals in this range there must exist an irrational (but  $\mathbb{Q} \cap [0,1]$  clearly contains no irrationals.)

2. Claim  $\sum_{n=0}^{\infty} a_n x^n$  and  $\sum_{n=1}^{\infty} n a_n x^{n-1}$  has same radius of convergence.

Proof This follows immediately from the fact that  $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$

implies

$$\limsup_{n \rightarrow \infty} \sqrt[n]{n|a_n|} = \limsup_{n \rightarrow \infty} n^{\frac{1}{n}} \sqrt[n]{|a_n|} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}. \quad \textcircled{*}$$

□

Verification of  $\textcircled{*}$ :

- The fact that  $n^{\frac{1}{n}} \sqrt[n]{|a_n|} \geq \sqrt[n]{|a_n|} \forall n \in \mathbb{N} \Rightarrow \limsup_{n \rightarrow \infty} n^{\frac{1}{n}} \sqrt[n]{|a_n|} \geq \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$
- Since for any  $\varepsilon > 0$   $n^{\frac{1}{n}} \leq (1+\varepsilon)$  eventually  
 $\Rightarrow \limsup_{n \rightarrow \infty} n^{\frac{1}{n}} \sqrt[n]{|a_n|} \leq (1+\varepsilon) \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} \quad \forall \varepsilon > 0$   
 $\Rightarrow \limsup_{n \rightarrow \infty} n^{\frac{1}{n}} \sqrt[n]{|a_n|} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ . □

3. (a) Since  $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \quad \forall |x| < 1 \Rightarrow \sum_{n=1}^{\infty} n4^{-n} = \underline{\underline{\frac{4}{9}}}$

$\uparrow$   
differentiate term-by-term

(b) Since  $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = -\log(1+x) \quad \forall |x| < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} 4^{-n} = \underline{\underline{\log(\frac{4}{5})}}$

$\uparrow$   
integration term-by-term

4. (a) (i)  $\frac{x^2}{2+x^2} = \frac{x^2}{2} \cdot \frac{1}{1+\frac{x^2}{2}} = \left(\frac{x^2}{2}\right)\left(1 - \frac{x^2}{2} + \left(\frac{x^2}{2}\right)^2 - \dots\right) \quad \text{if } |x| < \sqrt{2}$

$$= \underbrace{\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{8} - \dots}_{= P_6(x)} \quad \text{"the 6th order MacLaurin polynomial for } \frac{x^2}{2+x^2} \text{".}$$

(ii)  $f^{(6)}(0) = 6! \left(\frac{1}{8}\right) = \underline{\underline{90}}$

↑ coefficient in front of  $x^6$  above

(b) (i) Since  $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad \text{if } |x| < 1$

$$\Rightarrow \log x = \log(1+(x-1)) = \underbrace{(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots}_{= P_3(x)} \quad \text{if } |x-1| < 1$$

"the 3rd order Taylor poly for  $\log x$  centered at 1"

(ii) Since the Taylor series above is alternating when  $x > 1$

and the terms are decreasing when  $|x-1| < 1$  it follows from the Alternating Series Remainder Estimate that

$$|\log(1.5) - P(1.5)| \leq \frac{|1.5 - 1|^4}{4} = \frac{1}{4} \left(\frac{1}{2}\right)^4 = \underline{\underline{\frac{1}{64}}}$$

1st omitted term

(c) (i) Lagrangian Remainder Estimate for Maclaurin Series

If  $f$  is  $(n+1)$ -times differentiable on  $(-R, R)$ , then for any  $x \in (-R, R) \setminus \{0\}$   $\exists c$  between  $0$  &  $x$  such that

$$f(x) - [f(0) + f'(0)x + \dots + \frac{f^{(n)}(0)}{n!}x^n] = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

(ii) It follows from the Lagrangian Remainder Estimate that

$$\begin{aligned} |e^x - (1+x+\dots+\frac{x^n}{n!})| &= \frac{e^c}{(n+1)!}|x|^{n+1} \text{ for some } c \text{ between } 0 \text{ & } x, \\ &\leq \frac{e^{1/2}}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} \text{ if } |x| \leq \frac{1}{2} \\ &\leq \frac{1}{(n+1)!}\left(\frac{1}{2}\right)^n \text{ since } e^{1/2} \leq 2. \end{aligned}$$

Since  $\frac{1}{(n+1)!}\left(\frac{1}{2}\right)^n < \frac{1}{1000}$  if  $n=4$

$\Rightarrow 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$  approximate  $e^x$  to within  $10^{-3}$  for all  $|x| \leq \frac{1}{2}$ .

5. (a) We say that  $f$  is differentiable at  $x$  if

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

Claim

If  $f$  is differentiable at  $x_0$ , then  $f$  is continuous at  $x_0$

Proof Since

$$\begin{aligned}\lim_{x \rightarrow x_0} f(x) - f(x_0) &= \lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \\ &\quad \text{as } x \neq x_0 \\ &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \rightarrow x_0} (x - x_0) \\ &\quad \text{limit laws} \quad \text{(since both limits exist)} \\ &= f'(x_0) \cdot 0 = 0\end{aligned}$$

it follows that  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$  and hence that  $f$  is conts at  $x_0$ .  $\square$

(b) Let  $h(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

(i) Claim  $h$  is discontinuous at all  $x_0 \neq 0$ .

Proof

Let  $x_0 \neq 0$ . Since the rationals and irrationals are both dense in  $\mathbb{R}$  we know there exists

\* A seq  $\{x_n\}$  of rationals with  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ .

& \* A seq  $\{y_n\}$  of irrationals with  $y_n \rightarrow x_0$  as  $n \rightarrow \infty$

Since  $h(x_n) = x_n^2 \rightarrow x_0^2 \neq 0$  as  $n \rightarrow \infty$

&  $h(y_n) = 0 \quad \forall n \in \mathbb{N}$  so  $h(y_n) \rightarrow 0$  as  $n \rightarrow \infty$

it follows that  $h$  is discontinuous at  $x_0$ .  $\square$

(ii) Claim  $h$  is differentiable (and hence conts) at  $x_0 = 0$ .

Proof Since  $\left| \frac{h(x) - h(0)}{x - 0} \right| = \left| \frac{h(x)}{x} \right| \leq |x| \quad \forall x \neq 0$

it follow from the "squeeze theorem" that  $\lim_{x \rightarrow 0} \frac{h(x) - h(0)}{x - 0}$  exists and equals 0.  $\square$

- (iii) \*  $h$  is conts at  $x_0=0$  since  $h$  is diff'ble at  $x_0=0$  (Q5(a))
- \*  $h$  is not diff'ble at any  $x_0 \neq 0$  since  $h$  is not conts at any  $x_0 \neq 0$  (Q5(a) again)

### (c) Claim (Interior Extrema Theorem)

If  $f: [a, b] \rightarrow \mathbb{R}$  attains a minimum at a point  $c \in (a, b)$  and  $f'(c)$  exists, then  $f'(c) = 0$ .

Proof

We know that  $f(c) \leq f(x)$  for all  $x \in [a, b]$

Since  $f'(c)$  exists we know that for any seq  $\{x_n\}$  in  $[a, b] \setminus \{c\}$

with  $\lim_{n \rightarrow \infty} x_n = c$  it must be the case that

$$\lim_{n \rightarrow \infty} \frac{f(x_n) - f(c)}{x_n - c} = f'(c).$$

Thus, if we take any seq  $\{x_n\}$  in  $(a, c)$  with  $\lim_{n \rightarrow \infty} x_n = c$

$$\Rightarrow \frac{f(x_n) - f(c)}{x_n - c} \leq 0 \quad \forall n$$

and hence by order limit laws that  $f'(c) = \lim_{n \rightarrow \infty} \frac{f(x_n) - f(c)}{x_n - c} \leq 0$

While if we take any seq  $\{y_n\}$  in  $(c, b)$  with  $\lim_{n \rightarrow \infty} y_n = c$

$$\Rightarrow \frac{f(y_n) - f(c)}{y_n - c} \geq 0 \quad \forall n$$

and hence  $f'(c) = \lim_{n \rightarrow \infty} \frac{f(y_n) - f(c)}{y_n - c} \geq 0$ . Forces  $\underline{\underline{f'(c)=0}}$

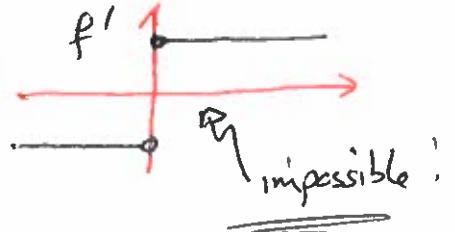
□

## 6. Claim (Darboux's Theorem)

If  $f: [a,b] \rightarrow \mathbb{R}$  is differentiable on  $[a,b]$  and  $f'(a) < 0 < f'(b)$ , then  $\exists c \in (a,b)$  with  $f'(c) = 0$ .

\*  $f'$  exists but may not be continuous, so we cannot apply the Intermediate Value Theorem!

This result is telling us that derivatives are "special", they have the "intermediate value property". In particular, it implies that not every function is the derivative of some other function, for example there is no function  $f$  for which  $f'$  has a jump discontinuity



Proof.

- \*  $f'(a) < 0 \Rightarrow \exists x_1 \in (a,b) \text{ s.t. } f(x_1) < f(a)$  [Why?]
- \*  $f'(b) > 0 \Rightarrow \exists x_2 \in (a,b) \text{ s.t. } f(x_2) < f(b)$  [Why?]

It follows that  $f$  attains a minimum on  $[a,b]$  at some point  $c \in (a,b)$

$Q5(c) \Rightarrow f'(c) = 0$  at this point.

□