

**Sample Exam 1 – Version 2**

*No calculators. Show your work. Give full explanations. Good luck!*

1. (8 points) Give counterexamples to the following **false** statements, no proofs are required.

*Note that in each instance the converse statement is in fact true.*

- (a) If  $\{x_n\}$  is bounded, then  $\{x_n\}$  is convergent.
  - (b) If  $\{x_n\}$  is convergent, then  $\{x_n\}$  is both bounded and monotone.
  - (c) If  $\{x_n\}$  contains a convergent subsequence, then  $\{x_n\}$  is bounded.
  - (d) If  $A$  contains its supremum, then  $A$  has finitely many elements.
2. (7 points) Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences of real numbers. Prove that if  $\lim_{n \rightarrow \infty} x_n = x$  and  $|x_n - y_n| \leq \frac{1}{n}$  for all  $n \in \mathbb{N}$ , then  $\lim_{n \rightarrow \infty} y_n = x$ .
  3. (8 points)
    - (a) Let  $\{x_n\}$  be a sequence of real numbers. Carefully state the definition of  $\lim_{n \rightarrow \infty} x_n = x$ .
    - (b) Use your definition to prove that  $\lim_{n \rightarrow \infty} \frac{3n+4}{n+1} = 3$ .
  4. (7 points)
    - (a) Carefully state the definition of a sequence of real numbers  $\{x_n\}$  being a Cauchy sequence.
    - (b) Give an example of a Cauchy sequence and prove that Cauchy sequences are always bounded.
  5. (10 points)
    - (a) Carefully state the *Axiom of Completeness* (the least upper bound axiom).
    - (b) Let  $\{x_n\}$  be a bounded increasing sequence of real numbers. Use the *Axiom of Completeness* to prove that  $\lim_{n \rightarrow \infty} x_n$  exists and equals  $\sup\{x_n : n \in \mathbb{N}\}$ .
  6. (10 points) Let  $\{x_n\}$  be a bounded sequence of real numbers.
    - (a) Carefully state the definition of  $\limsup_{n \rightarrow \infty} x_n$  and justify why it always exists for such sequences.
    - (b) Prove that if  $|x_n| \leq 10$  for all  $n \in \mathbb{N}$ , then  $\limsup_{n \rightarrow \infty} x_n \leq 10$ .

Math 3100 - Sample Exam 1 (Version 2) - SOLUTIONS

1. (a)  $x_n = (-1)^n$

(b)  $x_n = \frac{(-1)^n}{n}$

(c)  $x_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ odd} \\ n & \text{if } n \text{ even} \end{cases}$

(d)  $A = [0, 1]$ .

2. Claim

If  $\lim_{n \rightarrow \infty} x_n = x$  &  $|x_n - y_n| \leq \frac{1}{n} \ \forall n \in \mathbb{N}$ , then  $\lim_{n \rightarrow \infty} y_n = x$ .

Proof It follows from "Baby Squeeze" that  $(y_n - x_n) \rightarrow 0$ . Hence  
 $y_n = x_n + (y_n - x_n) \rightarrow x + 0 = x$  by limit laws.  $\square$

3. (a)  $\lim_{n \rightarrow \infty} x_n = x \Leftrightarrow \forall \varepsilon > 0 \ \exists N \text{ such that } n > N \text{ implies } |x_n - x| < \varepsilon$ .

(b) Claim  $\lim_{n \rightarrow \infty} \frac{3n+4}{n+1} = 3$

Proof let  $\varepsilon > 0$  & set  $N = \varepsilon^{-1}$ . It follows that if

$$n > N \Rightarrow \left| \frac{3n+4}{n+1} - 3 \right| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

4. (a)  $\{x_n\}$  Cauchy  $\Leftrightarrow \forall \varepsilon > 0 \ \exists N \text{ such that } n, m > N \Rightarrow |x_n - x_m| < \varepsilon$

(b)  $x_n = \frac{1}{n}$  is a Cauchy sequence (since it is convergent)

Claim

If  $\{x_n\}$  is Cauchy, then  $\{x_n\}$  is bounded.

Proof

Since  $\{x_n\}$  is Cauchy we know (taking  $\varepsilon=1$ ) that  $\exists N$  such that  $n, m \geq N \Rightarrow |x_n - x_m| < 1 \Rightarrow |x_n| = |x_m + (x_n - x_m)| \leq |x_m| + 1$ .

In particular,  $|x_n| \leq |x_{N+1}| + 1 \quad \forall n > N$  and hence

$$|x_n| \leq \max \{|x_1|, \dots, |x_N|, |x_{N+1}| + 1\} \quad \forall n \in \mathbb{N} \quad \square$$

5. (a) A.o.C  $\Leftrightarrow$  Every non-empty set of reals that is bounded above has a least upper bound.

(b) Claim (MKT)

If  $\{x_n\}$  is bounded increasing sequence of reals then

$\lim_{n \rightarrow \infty} x_n$  exists & equals  $\sup \{x_n : n \in \mathbb{N}\}$ .

Proof Since  $\{x_n\}$  is bounded the A.o.C ensures  $s = \sup \{x_n : n \in \mathbb{N}\}$  exist.

Let  $\varepsilon > 0$ . Since  $s = \sup \{x_n : n \in \mathbb{N}\} \exists N$  so that

$\{x_n\}$  increasing

$$s - \varepsilon < x_N \leq s$$

$$\Rightarrow s - \varepsilon < x_n \leq s < s + \varepsilon \quad \forall n > N$$

$$\Leftrightarrow |x_n - s| < \varepsilon \quad \forall n > N. \quad \square$$

6. Let  $\{x_n\}$  be a bounded sequence of reals.

(a)  $\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$ , where  $y_n = \sup \{x_n, x_{n+1}, \dots\}$ .

This is well-defined (always exists as a real number) since the AoC ensures  $y_n$  exists  $\forall n \in \mathbb{N}$  (since  $\{x_n\}$  is bounded), and defines a decreasing sequence (the fact that  $\{x_{n+1}, x_{n+2}, \dots\} \subseteq \{x_n, x_{n+1}, \dots\} \Rightarrow y_{n+1} \leq y_n \forall n \in \mathbb{N}$ ).

Moreover, since  $y_n \geq x_n \forall n \in \mathbb{N}$  and  $\{x_n\}$  is bounded ensures that  $\{y_n\}$  is bounded below, it thus follows from the MCT that  $\lim_{n \rightarrow \infty} y_n$  exists.

(b) Claim

If  $|x_n| \leq 10 \forall n \in \mathbb{N}$ , then  $\limsup_{n \rightarrow \infty} x_n \leq 10$

Proof

If  $|x_n| \leq 10 \forall n \in \mathbb{N}$ , then  $-10 \leq x_n \leq 10 \forall n \in \mathbb{N}$ .

It follows that 10 is also an upper bound for the set  $\{x_n, x_{n+1}, \dots\} \forall n \in \mathbb{N}$ .

Since  $y_n$  is the least upper bound for this set it follows that  $y_n \leq 10 \forall n \in \mathbb{N}$ . It then follows from the "order-limit law" that  $\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n \leq 10$  also.  $\square$