

Sample Exam 1 – Version 1

No calculators. Show your work. Give full explanations. Good luck!

1. (25 points)

- (a) Let $\{x_n\}$ be a sequence of real numbers. Carefully state the definition of the following:

- i. $\lim_{n \rightarrow \infty} x_n = x$
- ii. $\lim_{n \rightarrow \infty} x_n = \infty$.

- (b) Use the definition given in (i) to prove that $\lim_{n \rightarrow \infty} \frac{2n+1}{n-3} = 2$.

- (c) Use the definitions given above to prove that if $\lim_{n \rightarrow \infty} x_n = 2$, then

- i. $\{x_n\}$ is bounded
- ii. $\lim_{n \rightarrow \infty} \frac{1}{x_n} = \frac{1}{2}$
- iii. $\lim_{n \rightarrow \infty} (x_n + y_n) = \infty$ whenever $\lim_{n \rightarrow \infty} y_n = \infty$

2. (12 points)

- (a) Carefully state the *Monotone Convergence Theorem*.

- (b) Let $x_1 = 1$ and $x_{n+1} = \left(\frac{n}{n+1}\right)x_n^2$ for all $n \in \mathbb{N}$.

- i. Find x_2 , x_3 , and x_4 .
- ii. Show that $\{x_n\}$ converges and find the value of its limit.

3. (13 points) Let $\{x_n\}$ be a bounded sequence of real numbers.

- (a) Carefully state the definition of $\limsup_{n \rightarrow \infty} x_n$ and justify why it always exists for such sequences.

- (b) Prove that if $\alpha = \limsup_{n \rightarrow \infty} x_n$ and $\beta > \alpha$, then there exists an N such that $x_n < \beta$ whenever $n > N$.

- (c) Let $S = \{x : \text{there exists a subsequence of } \{x_n\} \text{ that converges to } x\}$.
- i. Why do we know that S is non-empty?
 - ii. Prove that if $x \in S$, then $x \leq \limsup_{n \rightarrow \infty} x_n$.

Math 3100 - Sample Exam 1 (Version 1) - SOLUTIONS

① (a)

(i) $\lim_{n \rightarrow \infty} x_n = x \Leftrightarrow \forall \varepsilon > 0 \exists N \text{ s.t. } n > N \Rightarrow |x_n - x| < \varepsilon$

(ii) $\lim_{n \rightarrow \infty} x_n = \infty \Leftrightarrow \forall M > 0 \exists N \text{ s.t. } n > N \Rightarrow x_n > M$

(b) Claim $\lim_{n \rightarrow \infty} \frac{2n+1}{n-3} = 2$

Proof Let $\varepsilon > 0$ & set $N = 3 + \frac{7}{\varepsilon}$. If $n > N$, then

$$\left| \frac{2n+1}{n-3} - 2 \right| = \left| \frac{7}{n-3} \right| = \frac{7}{n-3} < \varepsilon.$$

↑ since $n > 3 + \frac{7}{\varepsilon}$.

(c) (i) Claim: If $\lim_{n \rightarrow \infty} x_n = 2$, then $\{x_n\}$ is bounded.

Proof Since $\lim_{n \rightarrow \infty} x_n = 2 \exists N \text{ s.t. } n > N \Rightarrow |x_n - 2| < 1$
 $\Rightarrow |x_n| < 3 \text{ (if } n > N)$

Thus $|x_n| \leq \max \{|x_1|, \dots, |x_N|, 3\}$ for all $n \in \mathbb{N}$. \square

(ii) Claim: If $\lim_{n \rightarrow \infty} x_n = 2$, then $\lim_{n \rightarrow \infty} \frac{1}{x_n} = \frac{1}{2}$.

Proof Recall from above that since $\lim_{n \rightarrow \infty} x_n = 2 \exists N_1 \text{ s.t. } n > N_1 \Rightarrow |x_n| > 1$.

Let $\varepsilon > 0$. Since $\lim_{n \rightarrow \infty} x_n = 2 \exists N_2 \text{ s.t. } n > N_2 \Rightarrow |x_n - 2| < 2\varepsilon$

and hence if $n > \max\{N_1, N_2\} \Rightarrow \left| \frac{1}{x_n} - \frac{1}{2} \right| = \frac{|x_n - 2|}{2|x_n|} < \frac{|x_n - 2|}{2} < \varepsilon$.

\uparrow since $|x_n| > 1$ \uparrow since $|x_n - 2| < 2\varepsilon$ \square

(iii) Claim: If $\lim_{n \rightarrow \infty} x_n = 2$ & $\lim_{n \rightarrow \infty} y_n = \infty$, then $\lim_{n \rightarrow \infty} (x_n + y_n) = \infty$.

Proof

Recall that since $\lim_{n \rightarrow \infty} x_n = 2 \exists N_1 \text{ s.t. } n > N_1 \Rightarrow x_n > 1 > 0$.

Let $M > 0$. Since $\lim_{n \rightarrow \infty} y_n = \infty \exists N_2 \text{ s.t. } n > N_2 \Rightarrow y_n > M$

Hence if $n > \max\{N_1, N_2\} \Rightarrow x_n + y_n > y_n > M$. \square .

② (a) Monotone Convergence Theorem (MCT)

If $\{x_n\}$ is a bounded monotone sequence of reals, then

$\lim_{n \rightarrow \infty} x_n$ exists.

(b) Let $x_1 = 1$ & $x_{n+1} = \left(\frac{n}{n+1}\right)x_n^2 \forall n \in \mathbb{N}$.

$$(i) \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{1}{6}, \quad x_4 = \frac{1}{48}.$$

(ii) Claim: $\{x_n\}$ is decreasing and bounded below.

Proof

$$\bullet \quad x_1 = 1 \geq 0 \quad \& \quad x_n = \left(\frac{n}{n+1}\right)x_n^2 \geq 0 \quad \forall n \in \mathbb{N}.$$

$\rightarrow \{x_n\}$ bounded below by 0.

Subclaim: $x_n \leq 1 \quad \forall n \in \mathbb{N}$

[PF: $x_1 = 1 \leq 1 \checkmark$
Suppose $x_n \leq 1$ for some $n \in \mathbb{N}$.
It follows that $x_{n+1} = \left(\frac{n}{n+1}\right)x_n^2 \leq x_n^2 \leq 1$ \square]

\rightarrow In particular, $x_{n+1} \leq x_n^2 \leq x_n \quad \forall n \in \mathbb{N}$ \square
 \Rightarrow decreasing.

It now follows from MCT that $\lim_{n \rightarrow \infty} x_n = x$ for some $x \in \mathbb{R}$.

Since $x_{n+1} = \left(\frac{n}{n+1}\right)x_n^2 \quad \forall n \in \mathbb{N}$ it follows from limit laws

that $\downarrow \quad \downarrow \quad (\& \text{ uniqueness})$

$$x = (1)x^2 \Rightarrow x=0 \text{ or } X \leftarrow \text{since } \{x_n\} \text{ dec \&} \\ x_1 = 1. \quad \square$$

③ Let $\{x_n\}$ be a bounded sequence of reals.

$$(a) \limsup_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \sup \{x_n, x_{n+1}, \dots\} =: y_n$$

Since y_n exist for all $n \in \mathbb{N}$ (by A&C & fact that $\{x_n\}$ bdd) and defines a decreasing (since $\{x_{n+1}, x_{n+2}, \dots\} \subseteq \{x_n, x_{n+1}, \dots\}$) sequence of reals which is bounded below ($y_n \geq x_n \forall n$) it follows from the HCT that $\limsup_{n \rightarrow \infty} x_n$ exists.

(b) Claim: If $\alpha = \limsup_{n \rightarrow \infty} x_n$ & $\beta > \alpha$, then $\exists N$ s.t. $n > N \Rightarrow x_n < \beta$.

Proof

Let $y_n := \sup \{x_n, x_{n+1}, \dots\}$. Since $\lim_{n \rightarrow \infty} y_n = \alpha$ & $\beta > \alpha$

we know $\exists N$ s.t. $y_n < \beta$ (by taking $\varepsilon = \beta - \alpha$)

whenever $n > N$. Since $x_n \leq y_n \forall n \in \mathbb{N}$ it follows that

$$n > N \Rightarrow x_n < \beta. \quad \square$$

(c) Let $S = \{x : \exists \text{ subseq of } \{x_n\} \text{ that converges to } x\}$.

(i) Since $\{x_n\}$ is bounded it follows from the Bolzano-Weierstrass Theorem that $\{x_n\}$ contains a convergent subseq & hence S is nonempty.

(ii) Claim If $x \in S$, then $x \leq \alpha := \limsup_{n \rightarrow \infty} x_n$.

Proof: If $x > \alpha$, then $\exists \alpha < \beta < x$ & (b) $\Rightarrow \exists N$ such that if $n > N$ then $x_n < \beta$. In other words: $\{n : x_n \geq \beta\}$ is finite

But $x \in S \Rightarrow \exists \text{ subseq } \{x_{n_k}\}$ with $x_{n_k} \rightarrow x$. Hence $\forall \varepsilon > 0 \exists K$ such that $k > K$ implies $|x_{n_k} - x| < \varepsilon$. In other words: $\{K : |x_{n_k} - x| \leq \varepsilon\}$ infinite \Leftrightarrow In particular, with $\varepsilon = x - \beta$, we have $\{n : x_n \geq \beta\}$ is infinite $\Leftrightarrow \square$.