Math 3100 Assignment 3

Convergence of Sequences

Due at the beginning of class on Friday the 2nd of February 2018

1. What happens if we interchange or reverse the order of the quantifiers in the definition of convergence of a sequence?
   (a) *Definition:* A sequence $\{a_n\}$ *verconges* to a if *there exists* an $\varepsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n > N$ implies $|a_n - a| < \varepsilon$.
   Give an example of a vercongent sequence. Can you give an example a vercongent sequence that is divergent? What exactly is being described in this strange definition?
   (b) *Definition:* A sequence $\{a_n\}$ *conconges* to a if *there exists* a number N such that $n > N$ implies $|a_n - a| < \varepsilon$ *for all* $\varepsilon > 0$.
   Give an example of a concongent sequence. Can you give an example a concongent sequence that is divergent? What exactly is being described in this strange definition?

2. Verify the following using the definition of convergence of a sequence:
   (a) If $a_n \to a$, then $|a_n| \to |a|$. Is the converse true?
   (b) Let $a_n \geq 0$ for all $n \in \mathbb{N}$.
      i. Show that if $a_n \to 0$, then $\sqrt{a_n} \to 0$.
      ii. Show that if $a_n \to a$, then $\sqrt{a_n} \to \sqrt{a}$.
   (c) If $\{a_n\}$ is bounded (but not necessarily convergent) and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$.

3. Let $\{a_n\}$ be a convergent sequence with $\lim_{n\to\infty} a_n = a$. Prove the following two statements:
   (a) If $a_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq b$.
   (b) If $\{a_n\}$ is increasing, then $a_n \leq a$ for all $n \in \mathbb{N}$.

4. We say that $\{a_n\}$ *diverges to infinity*, and write $\lim_{n\to\infty} a_n = \infty$, if for every $M > 0$ there exists a number N such that $n > N$ implies that $a_n > M$.
   (a) Prove, using the definition above, that $\lim_{n\to\infty} n^p = \infty$ for all $p > 0$.
   (b) Prove that if $a_n > 0$ for all $n \in \mathbb{N}$, then
   $$\lim_{n\to\infty} a_n = \infty \quad \Longleftrightarrow \quad \lim_{n\to\infty} \frac{1}{a_n} = 0$$

5. (a) Let $x_1 = 3$ and $x_{n+1} = \dfrac{1}{4 - x_n}$ for all $n \in \mathbb{N}$.
      i. Show that $\{x_n\}$ is decreasing and satisfies $2 - \sqrt{3} \leq x_n \leq 3$ for all $n \in \mathbb{N}$.
      ii. Conclude that **if** the sequence $\{x_n\}$ converges, then it must converge to $2 - \sqrt{3}$.
   (b) Let $\{a_n\}$ be the Fibonacci sequence given recursively by $a_1 = 1$, $a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \in \mathbb{N}$. We now construct a new sequence $\{b_n\}$ by setting $b_n = a_{n+1}/a_n$ for all $n \in \mathbb{N}$.
      i. Show that $\{b_n\}$ satisfies the recursive formula $b_{n+1} = 1 + 1/b_n$ with $b_1 = 1$, and that $1 \leq b_n \leq 2$ for all $n \in \mathbb{N}$.
      ii. Conclude that **if** the sequence $\{b_n\}$ converges, then it must converge to $\dfrac{1+\sqrt{5}}{2}$.