Math 3100 **Spring 2017**

Exam 2

No calculators. Show your work. Give full explanations. Good luck!

1. (20 points)

   (a) Carefully state what it means to say that $\sum_{n=1}^{\infty} a_n$ converges to 2 and prove that if this indeed the case, then $\sum_{n=1}^{\infty}(10a_n)$ converges to 20.

   (b) Prove that if $b_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} b_n^2$ also converges.

   (c) Determine if the following series are absolutely convergent, conditionally convergent, or divergent. Justify your answers.

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}}$

   (d) Use the "Cauchy Condensation Test" to determine the convergence or divergence of

   $$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

   (e) Find all $x \in \mathbb{R}$ for which $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^{2n}}{n}$ converges.

2. (15 points) Let $f : \mathbb{R} \to \mathbb{R}$.

   (a) Carefully state the ε-δ definition of what it means for f to be *continuous* at x_0 and conclude that if f is continuous at x_0 with $f(x_0) = 2$, then there exists $\delta > 0$ such that $f(x) \geq 1$ whenever $|x - x_0| < \delta$.

   (b) Use the definition from part (a) to prove that $f(x) = \dfrac{1}{x}$ is continuous at $x_0 = 1$.

   (c) Prove that f is continuous at x_0 if and only if $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all sequences with $\lim_{n\to\infty} x_n = x_0$.

3. (15 points)

   (a) Carefully state what it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 and prove that if f is differentiable at x_0, then f is continuous at x_0.

   (b) Let $h(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.

      i. Prove that h is discontinuous at all $x \neq 0$.

      ii. Prove that h is differentiable at $x = 0$.

      iii. What can you say about the continuity of h at $x = 0$ and the differentiability of h at $x \neq 0$?

   (c) Let $f : [a, b] \to \mathbb{R}$.

      Prove that if f has a minimum at a point $c \in (a, b)$, and if $f'(c)$ exists, then $f'(c) = 0$.

4. (Bonus points) Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b).

   Prove that if $f'(a) < 0 < f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.