Sample Exam 2 – Version 2

No calculators. Show your work. Give full explanations. Good luck!

1. (15 points)

   (a) Carefully state the definition of what it means to say that $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

   (b) Use this definition to prove that if $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is divergent (not convergent), then $\sum_{n=1}^{\infty} (a_n + b_n)$ is divergent.

   (c) Prove that if $0 \leq a_n \leq c_n$ and $\sum_{n=1}^{\infty} c_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

2. (15 points)

   (a) Show that if $\lim_{n\to\infty} \sqrt{n} a_n = 2$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

   (b) Find all $x \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \dfrac{x^n}{n3^{n+1}}$ converges.

   (c) Find a sequence $\{a_n\}$ so that $\sum_{n=1}^{\infty} a_n x^n = \dfrac{4x}{2-x}$ for all $|x| < 2$.

3. (15 points)

   (a) Carefully state the ε-δ definition of what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous* at a point $x_0 \in \mathbb{R}$. Use this to show that $f(x) = \dfrac{2x+1}{x^2+1}$ is continuous at $x_0 = 2$.

   (b) Prove that if a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at x_0, then $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all sequences $\{x_n\}$ with $\lim_{n\to\infty} x_n = x_0$. Use this to show that

$$g(x) = \begin{cases} \cos(x^{-2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   is not continuous at $x_0 = 0$.

4. (5 points) Give examples of the following, no proofs are required:

   (a) A function that is continuous at 0 and discontinuous on $\mathbb{R} \setminus \{0\}$.

   (b) A series with bounded partial sums that is divergent.

   (c) Bonus Points:

   A sequence $\{b_n\}$ with $0 \leq b_n \leq \dfrac{1}{n}$ for each $n \in \mathbb{N}$, but for which $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ diverges.