Math 3100
Sample Exam 1 – Version 3

No calculators. Show your work. Give full explanations. Good luck!

1. (15 points) following limits or explain why it is divergent.

   (a) $\lim_{n\to\infty} \left(\frac{2n+1}{3-n}\right)^3$

   (b) $\lim_{n\to\infty} \left((-1)^n + \frac{1}{n}\right)$

   (c) $\lim_{n\to\infty} \frac{\cos(n)}{n^2}$

   (d) $\lim_{n\to\infty} \frac{n!+n}{2^n+3n!}$

   (e) $\lim_{n\to\infty} \frac{n+\log(n)}{n+1}$

2. (20 points)

   (a) Carefully state the definition of the convergence of a sequence $\{a_n\}$ to a real number L.

   (b) Using the definition of convergence, prove that

   $$\lim_{n\to\infty} \frac{5n+4}{2n-7} = 5/2.$$

3. (15 points)

   (a) Carefully state the definition of a sequence $\{a_n\}$ being bounded above.

   (b) Use this definition to prove that the sequence $a_n = \sqrt{n}$ is <u>not</u> bounded above.

4. (20 points) Assume that $\lim_{n\to\infty} b_n = L > 0$. Using <u>only</u> the definition of convergence prove the following two statements;

   (a) There exists $M > 0$ such that $|b_n| \le M$ for all $n \in \mathbb{N}$.

   (b) There exists some $N \in \mathbb{N}$ such that if $n > N$, then $b_n > L/2$.

5. (15 points) Prove that if $\{a_n\}$ is an increasing sequence with $\lim_{n\to\infty} a_n = L$, then $a_n \le L$ for all $n \in \mathbb{N}$.

6. **(Bonus points)** Let $\{a_n\}$ be the Fibonacci sequence given recursively by $a_1 = 1$ $a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for $n > 1$. We now construct a new sequence $\{b_n\}$ by setting $b_n = a_{n+1}/a_n$ for all $n \in \mathbb{N}$.

   (a) Show that $\{b_n\}$ satisfies the recursive formula $b_{n+1} = 1 + 1/b_n$ with $b_1 = 1$, and that $1 \le b_n \le 2$ for all $n \in \mathbb{N}$.

   (b) It follows from the formula in (a) that $b_{n+2} = 1 + \frac{b_n}{1+b_n}$ for all $n \in \mathbb{N}$, using this and induction prove that

      i. the subsequence $\{b_{2n}\}$ is decreasing

      ii. the subsequence $\{b_{2n-1}\}$ is increasing

   (c) Conclude that $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = \frac{1+\sqrt{5}}{2}$