Subsequences

Definition

Let $\{a_n\}$ be a sequence and $n_1 < n_2 < n_3 < \cdots$ be a strictly increasing sequence of natural numbers, then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots$$

is called a subsequence of $\{a_n\}_{n=1}^{\infty}$ and denoted by $\{a_{n_k}\}_{k=1}^{\infty}$.

Example

If $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ and $n_k = 2k$, then $\{a_{n_k}\}_{k=1}^{\infty}$ is the subsequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$ (the even indexed terms)

Theorem 1 (Inherited properties)

(a) Every subsequence of a bounded sequence is also bounded.

(b) Every subsequence of an increasing (decreasing) sequence is also increasing (decreasing)

(c) If $\lim_{n\to\infty} a_n = L$, then every subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ satisfies $\lim_{k\to\infty} a_{n_k} = L$ also.

(Every subsequence of a convergent sequence is also convergent, and converges to the <u>same</u> limit).

Corollary (of Theorem part (c)).

If a sequence contains two subsequences that converge to different limits, then the original sequence does not converge.

Examples: (1) Let $a_n = (-1)^n$.

Since the subsequence $a_{2k} = (-1)^{2k} = 1 \ \forall k \in \mathbb{N}$ converges to 1 & the subsequence $a_{2k-1} = (-1)^{2k-1} = -1 \ \forall k \in \mathbb{N}$ converges to -1, it follows that $\{a_n\}_{n\geq 1}^{\infty} = \{(-1)^n\}_{n\geq 1}^{\infty}$ is divergent

(2) Let $a_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ \frac{n}{2} & \text{if } n \text{ even}. \end{cases}$

So $\{a_n\} = \{1, 1, 1, 2, 1, 3, 1, 4, 1, 5, \ldots\}$.

Note that $\{a_n\}$ contains a subsequence that converges to 1, but no subsequence that converges to any other number. (It is however true that $\lim\limits_{k\to\infty} a_{2k} = \infty$, but ∞ is not a number!)

(*) We will see later than any bounded sequence that diverges must contain at least two subsequences that converge to different numbers.

Proof of Theorem 1

(a): If $\{a_n\}$ is bounded, then $\exists\ M \geq 0$ such that $|a_n| \leq M\ \forall n \in \mathbb{N}$

If $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, then clearly $|a_{n_k}| \leq M\ \forall k \in \mathbb{N}$ also.

(b): This is a homework problem.

(c): Suppose $\lim\limits_{n\to\infty} a_n = L$.

Let $\varepsilon > 0$ and $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Since $a_n \to L$ we know $\exists\ N$ such that if

$n > N$, then $|a_n - L| < \varepsilon$

Hence if $k > N$, then $n_k > N$ & $|a_{n_k} - L| < \varepsilon$ $\square$

since $\{n_k\}$ is strictly increasing!

We conclude this note with the following:

Rising Sun Lemma

Every sequence of real numbers contains a monotone subsequence.

* We make use of this result later, when we discuss the so-called "Bolzano-Weierstrass Theorem".

Proof of Rising Sun Lemma

Let $\{a_n\}$ be a sequence of real numbers.

⊛ We shall say the sequence has "a peak" at m if

$$a_m > a_n \quad \forall \ n > m.$$

Case 1: Suppose $\{a_n\}$ contains infinitely many peaks.

The the sequence $\{a_{n_k}\}_{k=1}^{\infty}$ with each n_k corresponding to a peak clearly forms a strictly decreasing subsequence

Case 2: Suppose $\{a_n\}$ contains only finitely many peaks

Now choose n_1 so that no peaks of $\{a_n\}$ occur at any m with $m \geq n_1$.

Since there is no peak at n_1, $\exists$ $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$

Since there is no peak at n_2, $\exists$ $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$

Continuing in this way we produce an increasing subsequence

$$a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \cdots$$

This completes the proof. $\square$