Special Limits

1. If $a_n \to 0$, then $a_n^p \to 0$ provided $p>0$.

   [In particular $\frac{1}{n^p} \to 0$ $\forall p>0$]

2. If $a_n \to a$ with $a \geq 0$, then $a_n^p \to a^p$ $\forall p>0$.

   [We have proved this when $p=\frac{1}{2}$ and $\forall p \in \mathbb{N}$
   directly from definition   using limit laws
   the general case will be proven after we discuss continuity.]

3. $\lim_{n\to\infty} r^n = 0$ if $|r|<1$.

   [Proven (using Binomial Thm) at end of "Baby Squeeze" notes]

4. $\lim_{n\to\infty} \frac{n^p}{x^n} = 0$ if $|x|>1$ and $p \in \mathbb{R}$.
   (only interesting if $p>0$!)

   [See below for proof]

5. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ $\forall x \in \mathbb{R}$ (only interesting for $|x|>1$)

   [Easy application of "Ratio Test"]

6. $\lim_{n\to\infty} \frac{\log(n)}{n^p} = 0$ $\forall p>0$

   [We will prove this later in the course, after differentiation]

7. $\lim_{n\to\infty} x^{1/n} = 1 \quad \forall\, x > 0$.

8. $\lim_{n\to\infty} n^{1/n} = 1$

[See below for proof of these, see also MCT notes for an alternative proof of 7.]

9. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

[We will discuss this limit later in the course.].

Application of Limit Laws using these special limits.

① Evaluate the following limits or explain why they diverge.

(a) $\lim_{n\to\infty} \dfrac{2n! - n}{3^n + 7n!}$

(b) $\lim_{n\to\infty} \dfrac{n^2 \cos(n)}{2^n}$

(c) $\lim_{n\to\infty} \left(\dfrac{\sqrt{n}}{3^n} + (-1)^n \right)$.

(d) $\lim_{n\to\infty} \dfrac{(-1)^n 3^n}{n^n (n+1)}$

limit laws & "special limit 5"

(a): Since $\dfrac{2n!-n}{3^n+7n!} = \dfrac{2-\frac{1}{(n-1)!}}{\frac{3^n}{n!}+7} \longrightarrow \dfrac{2-0}{0+7} = \dfrac{2}{7}$.

it follows that $\dfrac{2n!-n}{3^n+7n!} \to \dfrac{2}{7}$.

"special limit 4"

(b): Since $\left|\dfrac{n^2\cos(n)}{2^n}\right| \leq \dfrac{n^2}{2^n}$ and $\dfrac{n^2}{2^n} \longrightarrow 0$

it follows from "Baby Squeeze" that $\dfrac{n^2\cos(n)}{2^n} \to 0$.

"Special limit 4"

(c): Let $a_n = \dfrac{\sqrt{n}}{3^n} + (-1)^n$. Note that $\dfrac{\sqrt{n}}{3^n} \longrightarrow 0$.

If $\lim a_n$ exists, then it would follow from the "sum limit law" that $(-1)^n = \underbrace{a_n - \frac{\sqrt{n}}{3^n}}$ would also be convergent

difference of two convergent sequences

But $(-1)^n$ is divergent, so $\{a_n\}$ must be too.

(d): Let $a_n = \dfrac{(-1)^n 3^n}{n^n(n+1)}$. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}3^{n+1}}{(n+1)^{n+1}(n+2)} \cdot \frac{n^n(n+1)}{(-1)^n 3^n}\right| = 3\frac{1}{n+2}\cdot\left(\frac{n}{n+1}\right)^n$$

$$\longrightarrow 3(0)\left(\frac{1}{e}\right) = 0 < 1$$

limit 9 above

it follows from the "Ratio Test" that $\lim\limits_{n\to\infty} a_n = 0$.

Some Proofs of Special Limits

Claim 1 $\lim_{n\to\infty} \frac{n^p}{x^n} = 0$ if $|x|>1$ and $p>0$.

Proof By "Special Limit 1" it suffices to show that

$$(*) \quad \lim_{n\to\infty} \frac{n}{y^n} = 0 \quad \forall\, y>1$$

[Since (letting $y = |x|^{1/p}$) we would then get $\frac{n^p}{|x|^n} \to 0$]

But $(*)$ follows immediately from the "Ratio Test". $\square$

Claim 2 $\lim_{n\to\infty} x^{1/n} = 1 \quad \forall\, x>0$.

Proof Suppose $x>1$, then $y := x^{1/n} - 1 \geq 0$.

Since $x = (1+y)^n \geq ny$ (by Binomial Thm)

$$\Rightarrow 0 \leq y \leq \frac{x}{n}$$

Since $\frac{x}{n} \to 0$ it follow "Baby Squeeze" that

$$y \to 0 \Leftrightarrow x^{1/n} \to 1.$$

Since $a_n \to 1 \Leftrightarrow \frac{1}{a_n} \to 1$ (limit laws)

the result also follows when $0<x<1$. $\square$

Claim 3: $\lim_{n\to\infty} n^{1/n} = 1$

Binomial Thm

Proof Let $y = n^{1/n} - 1 \geq 0$. Since $n = (1+y)^n \geq \frac{n(n+1)}{2} y^2$

$\Rightarrow 0 \leq y \leq \sqrt{\frac{2}{n+1}}$. Since $\sqrt{\frac{2}{n+1}} \to 0$ result follows by "Baby Squeeze".