Math 3100 **Fall 2018**

Infinite Series

1. Important infinite series

Geometric series: $\sum_{n=0}^{\infty} r^n$ converges $\iff$ $|r| < 1$. If $|r| < 1$, then $\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r}$.

The p-series: $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges $\iff$ $p > 1$.

2. Definition and Properties of Convergent Series

Definition. Given a sequence $\{a_n\}$ we let $s_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$ denote its nth partial sum.

If $\{s_n\}$ converges we define

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n$$

and say that the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent (or that the original sequence $\{a_n\}$ is summable).

If $\{s_n\}$ diverges we say that the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 1 (Manipulation of Convergent Series). *If $\{a_n\}$ and $\{b_n\}$ are two summable sequences and $c \in \mathbb{R}$, then the sequences $\{a_n + b_n\}$ and $\{c\, a_n\}$ are also summable with*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \quad \text{and} \quad \sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n.$$

Theorem 2. *If $\{a_n\}$ is a summable sequence, that is if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Remark 1: This gives us the following "Test for Divergence": If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Remark 2: Warning! The converse of Theorem 2 is FALSE, in other words $\lim\limits_{n\to\infty} a_n = 0$ does not in and of itself guarantee $\sum_{n=1}^{\infty} a_n$ converges. Consider for example the so-called "harmonic series" $\sum_{n=1}^{\infty} \frac{1}{n}$.

Theorem 2 can either be verified directly from the definition (and limit laws) or deduced from the following

Theorem 3 (Cauchy Criterion).

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \iff \quad \text{for every } \varepsilon > 0, \text{ there exists } N \text{ such that } \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon \text{ if } n > m > N.$$

3. Convergence Tests for Series of non-negative terms

Theorem 4 (Monotone Convergence Theorem on Series). *If $a_n \geq 0$ and $s_n = a_1 + \cdots + a_n$, then*

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \iff \quad \{s_n\} \quad \text{bounded.}$$

Theorem 5 (Cauchy Condensation Test). *If $\{a_n\}$ is a decreasing sequence of non-negative terms, then*

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \iff \quad \sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots \quad \text{converges.}$$

This test is only really used to establish p-series and its close relatives.

Theorem 6 (Direct Comparison Test). *Suppose $0 \leq a_n \leq b_n$ for all sufficiently large $n \in \mathbb{N}$.*

(i) *If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

(ii) *If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.*

[*If $0 \leq a_n \leq b_n$ holds for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges, then one can conclude that $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.*]

Corollary 7 (Limit Comparison Test)**.** *If $a_n \geq 0$ and $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then*

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} b_n \quad \text{converges.}$$

4. Series of both negative and non-negative terms

Theorem 8 (Absolute Convergence implies Convergence)**.**

$$\text{If} \sum_{n=1}^{\infty} |a_n| \text{ converges, then} \sum_{n=1}^{\infty} a_n \text{ converges.}$$

This can be deduced as a consequence of either Theorem 3 or Theorem 4. The statement can, in fact, be shown to be equivalent to (and hence is yet another formulation of) the Axiom of Completeness.

Theorem 9 (Alternating Series Test)**.** *If $\{b_n\}$ is decreasing with limit 0, then $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges and the error obtained by "cutting off" the infinite series after N terms, namely*

$$\left|\sum_{n=1}^{N}(-1)^{n+1}b_n - \sum_{n=1}^{\infty}(-1)^{n+1}b_n\right| \leq b_{N+1}.$$

Theorem 10 (Ratio Test – A Computational Tool)**.** *Let $\{a_n\}$ be a sequence of non-zero terms.*

- *If $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, so in particular if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty}|a_n|$ converges.*
- *If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ infinitely often, so in particular if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Recall, by considering $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$, that the Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Theorem 11 (Root Test – Mainly a Theoretical Tool)**.** *Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.*

- *If $\alpha < 1$, then $\sum_{n=1}^{\infty}|a_n|$ converges.*
- *If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Recall, again by considering for example $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$, that the Root Test is inconclusive if $\alpha = 1$.

Corollary 12 (Convergence of Power Series)**.** *The domain of convergence for a power series $\sum_{n=1}^{\infty} c_n x^n$ is either $\{0\}$, all of $\mathbb{R}$, or precisely one of $(-R, R)$, $(-R, R]$, $[-R, R)$, or $[-R, R]$ for some $R > 0$.*

This follows directly from the Theorem 11 together with the fact that $\limsup_{n\to\infty} \sqrt[n]{|c_n x^n|} = |x| \limsup_{n\to\infty} \sqrt[n]{|c_n|}$.

5. Strategy for analyzing $\sum_{n=1}^{\infty} a_n$

1. Does $a_n \to 0$?

   If NO, then $\sum_{n=1}^{\infty} a_n$ diverges.
2. Does $\sum_{n=1}^{\infty} |a_n|$ converge?

   If YES, then $\sum_{n=1}^{\infty} a_n$ converges absolutely, and hence converges. Try using
   - geometric series and p-series
   - "direct" or "limit" comparison tests
   - ratio (or root) test
   - Cauchy condensation test (or integral test if you are familiar with that)
3. If $\sum_{n=1}^{\infty} |a_n|$ does not converge or you cannot decide, then try
   - alternating series test

     If this test applies, then $\sum_{n=1}^{\infty} a_n$ converges.

   Recall that if

   $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say $\sum_{n=1}^{\infty} a_n$ converges conditionally.