Theorem (Ratio Test for Sequences)

If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L < 1$, then $\lim\limits_{n\to\infty} a_n = 0$.

Applications

Claim 1: $\lim\limits_{n\to\infty} \dfrac{n}{2^n} = 0$

Proof Let $a_n = \dfrac{n}{2^n}$. Since $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{n+1}{2^{n+1}} \cdot \dfrac{2^n}{n}\right|$

$$= \left(\frac{n+1}{n}\right)\frac{1}{2} \to \frac{1}{2} < 1$$

it follows from "Ratio Test" that $\dfrac{n}{2^n} \to 0$. $\square$

Claim 2 $\lim\limits_{n\to\infty} \dfrac{(-3)^n}{n!} = 0$

Proof Let $a_n = \dfrac{(-3)^n}{n!}$.

Since $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(-3)^{n+1}}{(n+1)!} \cdot \dfrac{n!}{(-3)^n}\right| = 3\dfrac{1}{n+1} \to 0 < 1$

it follows from "Ratio Test" that $\dfrac{(-3)^n}{n!} \to 0$ $\square$

(*) These are examples of a more general phenomenon, namely

FACT 1: If $|x| > 1$ and $p \in \mathbb{R}$, then $\lim\limits_{n\to\infty} \dfrac{n^p}{x^n} = 0$.

The proofs of these facts are discussed later — see "special limits"

FACT 2: If $x \in \mathbb{R}$, then $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$.

Proof of Ratio Test for Sequences

Choose $L < r < 1$. Since $\left|\frac{a_{n+1}}{a_n}\right| \to L$ we know

$\exists N$ such that $n > N$ implies $|a_{n+1}| < r|a_n|$.

In particular,

$$|a_{N+2}| < r|a_{N+1}|$$

$$|a_{N+3}| < r|a_{N+2}| < r^2|a_{N+1}|$$

$$|a_{N+4}| < r|a_{N+3}| < r^3|a_{N+1}|$$

$$\vdots$$

and in general if $n > N$, then

$$|a_n| < r^n \underbrace{\left(r^{-(N+1)}|a_{N+1}|\right)}_{\text{this is a constant!}}$$

Since $r^n\left(r^{-(N+1)}|a_{N+1}|\right) \longrightarrow 0$ as $n \to \infty$

(since $0 < r < 1$) it follows from "Baby Squeeze"

that $\lim\limits_{n\to\infty} a_n = 0$. $\square$