Two Partial Converses to the fact that "Convergence ⇒ Bounded"

Theorem 1 (Monotone Convergence Theorem (MCT))

If a sequence is monotone and bounded, then it converges

Theorem 2 (Bolzano - Weierstrass (BW))

If a sequence is bounded, then it contains a convergent subsequence.

* It is important in both theorems above that the sequence in question is a sequence of real numbers, their proofs use the "completeness of $\mathbb{R}$".

Proof of MCT

Let $\{a_n\}$ be a bounded increasing sequence of reals. (the proof for decreasing is similar). To prove that $\{a_n\}$ is convergent, using the definition of convergence, we are going to need a candidate for the limit. Consider the set

$$A = \{a_n : n \in \mathbb{N}\}.$$

By assumption this set is bounded (and non-empty), so by the AoC we can let $s = \sup A$.

It seems reasonable to claim that $\lim_{n\to\infty} a_n = s$.

To prove this, let $\varepsilon > 0$. Since $s = \sup A = \sup\{a_n : n \in \mathbb{N}\}$ we know $\exists$ element a_N in the sequence such that

$$s - \varepsilon < a_N \le s .$$

The fact that $\{a_n\}$ is increasing ensures that if $n > N$, then

$$s - \varepsilon < a_n \le s < s + \varepsilon$$

$$\Rightarrow \quad |a_n - s| < \varepsilon . \qquad \square$$

<u>Note</u>: We in fact prove that if $\{a_n\}$ is increasing & bounded, then

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\} .$$

It is also true that if $\{a_n\}$ is decreasing & bounded, then

$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\} .$$

<u>Proof of BW</u>

Let $\{a_n\}$ be a bounded sequence of reals. It follows from the "Rising Sun Lemma" that $\{a_n\}_{n=1}^{\infty}$ contains a <u>bounded</u> and <u>monotone</u> subsequence $\{a_{n_k}\}_{k=1}^{\infty}$. The MCT then implies that this subsequence converges. $\square$