

Infinite Series

Given a sequence $\{a_n\}$ of real numbers, we define the corresponding "sequence of n^{th} partial sums" $\{s_n\}$ by

$$s_n := a_1 + a_2 + \dots + a_n \quad \text{for all } n \in \mathbb{N}.$$

Note: In "sigma notation" $s_n = \sum_{k=1}^n a_k$.

⊛ We say that the infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

converges if $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$ exists.

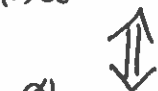
[We say $\sum_{n=1}^{\infty} a_n$ converges to A , and write $\sum_{n=1}^{\infty} a_n = A$, if $\lim_{n \rightarrow \infty} s_n = A$]

• If $\sum_{n=1}^{\infty} a_n$ does not converge we say it diverges.

Examples

1. If $a_n = 1 \quad \forall n \in \mathbb{N}$, then $s_n = n \quad \forall n \in \mathbb{N}$ and hence

$$\lim_{n \rightarrow \infty} s_n = \infty.$$



$$\sum_{n=1}^{\infty} 1 \quad \underline{\text{DIVERGES}}.$$

2. If $a_n = (-1)^{n-1} \forall n \in \mathbb{N}$, then

$$S_1 = 1$$

$$S_2 = 1 - 1 = 0$$

$$S_3 = 1 - 1 + 1 = 1$$

⋮

$$S_n = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd.} \end{cases}$$

It clearly contains two subsequences that converge to different limits (0 & 1)

Since $\lim_{n \rightarrow \infty} S_n$ does not exist we conclude that

$$\sum_{n=1}^{\infty} (-1)^{n-1} \text{ DIVERGES .}$$

3. If $a_n = \frac{1}{n(n+1)}$ for all $n \in \mathbb{N}$, then (using the fact that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$) we see that

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) = 1 - \frac{1}{3}$$

$$S_3 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) = 1 - \frac{1}{4}$$

⋮

$$\underline{S_n = 1 - \frac{1}{n+1} \quad \forall n \in \mathbb{N} .}$$

← Easily proved by induction.

Since $\lim_{n \rightarrow \infty} S_n = 1$ we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \text{ converges to 1}$$

⊗ This is an example of what is called a "telescoping series"

4. If $a_n = \frac{1}{2^{n-1}} \forall n \in \mathbb{N}$, then

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$$

⋮

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} \quad \forall n \in \mathbb{N}.$$

Since $\frac{1}{2} S_n = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} + \frac{1}{2^n}$ it follows that

$$S_n - \frac{1}{2} S_n = 1 - \frac{1}{2^n}$$

and hence that

$$S_n = \frac{1 - \frac{1}{2^n}}{\frac{1}{2}} = 2 - \frac{1}{2^{n-1}}$$

Since $\lim_{n \rightarrow \infty} S_n = 2$ we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \text{ converges to } 2.$$

⊗ This is a special case of what is referred to as a Geometric Series.

Geometric Series (The "Mother Example")

- $\sum_{n=1}^{\infty} r^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.
- Moreover, if $|r| < 1$, then $\sum_{n=1}^{\infty} r^{n-1}$ converges to $\frac{1}{1-r}$.

Proof

- Examples 1 & 2 already deal with the case when $|r| = 1$.
- Suppose $|r| \neq 1$, then

$$S_n = 1 + r + r^2 + \dots + r^{n-1} \quad \forall n \in \mathbb{N}$$

$$\& \quad r S_n = r + r^2 + \dots + r^{n-1} + r^n \quad \forall n \in \mathbb{N}$$

$$\Rightarrow (1-r)S_n = 1 - r^n \quad \forall n \in \mathbb{N}$$

and hence, since $|r| \neq 1$, that $S_n = \frac{1-r^n}{1-r} \quad \forall n \in \mathbb{N}$.

- Since $\lim_{n \rightarrow \infty} r^n = 0$ if $|r| < 1$ & DNE if $|r| > 1$
we may conclude that $\lim_{n \rightarrow \infty} S_n = \frac{1}{1-r}$ if $|r| < 1$
& diverges if $|r| > 1$.

This is of course precisely what it means to say

$$\sum_{n=1}^{\infty} r^{n-1} \text{ converges to } \frac{1}{1-r} \text{ if } |r| < 1 \text{ \& } \sum_{n=1}^{\infty} r^{n-1} \text{ diverges if } |r| > 1$$