Infinite Series

Given a sequence $\{a_n\}$ of real numbers, we define the corresponding "sequence of n^{th} partial sums" $\{s_n\}$ by

$$s_n := a_1 + a_2 + \cdots + a_n \quad \text{for all } n \in \mathbb{N}.$$

Note: In "sigma notation" $s_n = \sum_{k=1}^{n} a_k$.

(*) We say that the infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

converges if $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ exists.

[We say $\sum_{n=1}^{\infty} a_n$ converges to A, and write $\sum_{n=1}^{\infty} a_n = A$, if $\lim_{n\to\infty} s_n = A$]

- If $\sum_{n=1}^{\infty} a_n$ does not converge we say it diverges

Examples

1. If $a_n = 1 \; \forall n \in \mathbb{N}$, then $s_n = n \; \forall n \in \mathbb{N}$ and hence

$$\lim_{n\to\infty} s_n = \infty.$$

$$\Updownarrow$$

$$\sum_{n=1}^{\infty} 1 \quad \text{DIVERGES}.$$

2. If $a_n = (-1)^{n-1}\ \forall n \in \mathbb{N}$, then

$$S_1 = 1$$
$$S_2 = 1 - 1 = 0$$
$$S_3 = 1 - 1 + 1 = 1$$
$$\vdots$$
$$S_n = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd.} \end{cases}$$

It clearly contains two subsequences that converge to different limits (0 & 1)

Since $\lim_{n\to\infty} S_n$ does not exist we conclude that

$\sum_{n=1}^{\infty} (-1)^{n-1}$ DIVERGES.

3. If $a_n = \frac{1}{n(n+1)}$ for all $n \in \mathbb{N}$, then (using the fact that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$) we see that

$$S_1 = 1 - \tfrac{1}{2}$$
$$S_2 = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) = 1 - \tfrac{1}{3}$$
$$S_3 = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = 1 - \tfrac{1}{4}$$
$$\vdots$$
$$S_n = 1 - \frac{1}{n+1} \quad \forall n \in \mathbb{N}.$$

Easily proved by induction.

Since $\lim_{n\to\infty} S_n = 1$ we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \text{ converges to } 1$$

(*) This is an example of what is called a "telescoping series"

4. If $a_n = \frac{1}{2^{n-1}}$ $\forall n \in \mathbb{N}$, then

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2}$$
$$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$$
$$\vdots$$
$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \quad \forall n \in \mathbb{N}.$$

Since $\frac{1}{2} S_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}$ it follows that

$$S_n - \frac{1}{2} S_n = 1 - \frac{1}{2^n}$$

and hence that

$$S_n = \frac{1 - \frac{1}{2^n}}{\frac{1}{2}} = 2 - \frac{1}{2^{n-1}}$$

Since $\lim_{n \to \infty} S_n = 2$ we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \text{ converges to } 2.$$

(*) This is a special case of what is refered to as a Geometric Series.

Geometric Series (The "Mother Example")

- $\sum_{n=1}^{\infty} r^{n-1}$ converges if $|r|<1$ and diverges if $|r|\geq 1$.
- Moreover, if $|r|<1$, then $\sum_{n=1}^{\infty} r^{n-1}$ converges to $\frac{1}{1-r}$.

Proof

- Examples 1 & 2 already deal with the case when $|r|=1$.
- Suppose $|r|\neq 1$, then

$$S_n = 1+r+r^2+\cdots+r^{n-1} \quad \forall n\in\mathbb{N}$$

$$\& \quad rS_n = r+r^2+\cdots+r^{n-1}+r^n \quad \forall n\in\mathbb{N}$$

$$\Rightarrow \quad (1-r)s_n = 1-r^n \quad \forall n\in\mathbb{N}$$

and hence, since $|r|\neq 1$, that $S_n = \frac{1-r^n}{1-r} \quad \forall n\in\mathbb{N}$.

- Since $\lim_{n\to\infty} r^n = 0$ if $|r|<1$ & DNE if $|r|>1$

we may conclude that $\lim_{n\to\infty} S_n = \frac{1}{1-r}$ if $|r|<1$

& diverges if $|r|>1$.

This is of course precisely what it means to say
$\sum_{n=1}^{\infty} r^{n-1}$ converges to $\frac{1}{1-r}$ if $|r|<1$ & $\sum_{n=1}^{\infty} r^{n-1}$ diverges if $|r|>1$