# Cauchy Sequences

The following definition bears a striking resemblance to the definition of convergence of a sequence.

## Definition

A sequence $\{a_n\}$ is called Cauchy if $\forall \varepsilon > 0 \ \exists N$ such that if $n, m > N$, then $|a_n - a_m| < \varepsilon$.

Informally, a sequence is Cauchy if its terms are eventually all close to each other. One should contrast this with the notion of a sequence being convergent, which says that its terms are eventually all close to some limit value.

Spoiler: Convergent sequences are Cauchy and Cauchy sequences are convergent!

The significance of the definition of Cauchy is that there is no mention of a limit. This is somewhat like the situation with the MCT in that we will have another way to prove a sequence converges without knowing what the limit might be.

## Theorem 1

Every convergent sequence is a Cauchy sequence.

## Proof

Let $\{a_n\}$ be a sequence with $\lim_{n\to\infty} a_n = L$.

Let $\varepsilon > 0$. Since $a_n \to L$ we know $\exists N$ such that if $n > N$ then $|a_n - L| < \varepsilon/2$ (since $\varepsilon/2 > 0$).

Hence if $n, m > N$, then

$$|a_n - a_m| = |(a_n - L) + (L - a_m)|$$

since $n > N$ & $m > N$

$$\leq |a_n - L| + |a_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

## Lemma

If $\{a_n\}$ is Cauchy, then $\{a_n\}$ is bounded.

## Proof

Given $\varepsilon = 1$, $\exists N$ such that $n, m > N$ implies $|a_n - a_m| < 1$. In particular, taking $m = N+1$, we see that

$$|a_n| = |a_{N+1} + (a_n - a_{N+1})| < |a_{N+1}| + 1 \quad \forall n > N,$$

and hence that $|a_n| \leq \max\{|a_1|, |a_2|, \ldots, |a_N|, |a_{N+1}| + 1\}$

$\forall n \in \mathbb{N}$ $\square$

**Theorem** (Cauchy Criterion (CC))

A sequence converges if and only if it is Cauchy

$\Longleftrightarrow$

**Proof**

($\Rightarrow$) This is the content of Thm 1 above.

($\Leftarrow$) Let $\{a_n\}$ be Cauchy. It follows from the Lemma above that $\{a_n\}$ is bounded and hence, from the Bolzano-Weierstrass theorem, that $\{a_n\}$ contains a subsequence $\{a_{n_k}\}$ which is convergent. Set

$$L = \lim_{k\to\infty} a_{n_k}.$$

We will now show that $\lim_{n\to\infty} a_n = L$.

Let $\varepsilon > 0$. Since $\{a_n\}$ is Cauchy we know $\exists N$ such that if $n, m > N$ then $|a_n - a_m| < \varepsilon/2$ (since $\varepsilon/2 > 0$).

We also know that $a_{n_k} \to L$, so in particular there is an element of this subsequence $a_{n_K}$ with $n_K > N$ & $|a_{n_K} - L| < \frac{\varepsilon}{2}$ (since $\varepsilon/2 > 0$).

$$\Rightarrow |a_n - L| \leq |a_n - a_{n_K}| + |a_{n_K} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{if } n > N$$

$\square$