# Two Applications of the MCT

**Claim 1** If $a_1 = \sqrt{3}$ and $a_{n+1} = \sqrt{3+2a_n}$ $\forall n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n = 3$.

**Proof**

- We first show, by induction, that $\sqrt{3} \le a_n \le 3$ $\forall n \in \mathbb{N}$:

  Base Case $(n=1)$: $\sqrt{3} \le \sqrt{3} \le 3$ ✓

  Suppose $\sqrt{3} \le a_n \le 3$ for some $n \in \mathbb{N}$, then

  $$3 \le 3 + 2a_n \le 9 \Rightarrow \sqrt{3} \le \underbrace{\sqrt{3+2a_n}}_{=a_{n+1}.} \le 3 \quad ✓$$

- We now show that $\{a_n\}$ is increasing:

  $$a_{n+1} - a_n = \sqrt{3+2a_n} - a_n$$

  $$= \frac{3 + 2a_n - a_n^2}{\sqrt{3+2a_n} + a_n}$$

  $$= \underbrace{-}_{<0} \frac{\overbrace{(a_n - 3)}^{\le 0}\overbrace{(a_n+1)}^{>0}}{\underbrace{\sqrt{3+2a_n} + a_n}_{>0}} \ge 0 \quad \text{since } \sqrt{3} \le a_n \le 3 \ \forall n \in \mathbb{N}.$$

- MCT $\Rightarrow \lim_{n\to\infty} a_n = L$ for some $\sqrt{3} \le L \le 3$.

  Order limit Laws

  Since $a_{n+1} = \sqrt{3+2a_n}$ $\forall n \in \mathbb{N}$

  $x_n \to x \Rightarrow \sqrt{x_n} \to \sqrt{x}$ if $x_n \ge 0$ $\forall n$

  and $a_{n+1} \to L$ & $\sqrt{3+2a_n} \to \sqrt{3+2L}$

  Uniqueness of limits $\Rightarrow L = \sqrt{3+2L}$

  $$\Rightarrow L^2 - 2L - 3 = 0 \Rightarrow \underline{\underline{L = 3}} \text{ or } \cancel{L = -1}$$

$\square$

Claim 2  If $x > 1$, then $\lim_{n\to\infty} x^{1/n} = 1$.

Proof

Since $x^{1/n} \geq 1 \ \forall n \in \mathbb{N}$ & $x^{1/n+1} \leq x^{1/n} \ \forall n \in \mathbb{N}$

it follows from the MCT that $\lim_{n\to\infty} x^{1/n} = L$ for some $L \geq 1$.

Since $x^{1/n} \longrightarrow L \Rightarrow \sqrt{x^{1/n}} = x^{1/2n} \longrightarrow \sqrt{L}$

But $\{x^{1/2n}\}$ is a subsequence of $\{x^{1/n}\}$ so $x^{1/2n} \to L$.

By uniqueness of limits it must be true that $L = \sqrt{L}$

$\Rightarrow \cancel{L=0}$ or $L = 1$ $\square$

Exercise

Use the MCT to give a new proof that if $|x| < 1$, then

$$\lim_{n\to\infty} x^n = 0 .$$