Least upper bounds, Greatest lower bounds, and the Axiom of Completeness

Axiom of Completeness (AoC)

Every non-empty set of real numbers that is bounded above has a least upper bound.

* This is the defining property of $\mathbb{R}$ that distinguishes it from $\mathbb{Q}$, but what does it mean? ¿What is a least upper bound?

Definition (Least Upper Bound)

A real number s is the least upper bound for a set $A \subseteq \mathbb{R}$ if it satisfies the following:

(i) $a \leq s \ \forall a \in A$ (s is an upper bound for A)

(ii) $\forall \varepsilon > 0 \ \exists a \in A$ with $s - \varepsilon < a$

(any number less than s is not an upper bound for A)

* The least upper bound is also frequently called the supremum of A and denoted by sup(A).

Examples:

1. If $A = [0,1)$, then $\sup A = 1$
2. If $A = [0,1] \cup \{2\}$, then $\sup A = 2$
3. If $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$, then $\sup A = 1$.

Verification of Example 3

- Since $1-\frac{1}{n}\le 1$ $\forall n\in\mathbb{N}$, 1 is an upper bound for A.
- Let $\varepsilon>0$. Since $\exists n\in\mathbb{N}$ such that $0<\frac{1}{n}<\varepsilon$

$$\Rightarrow 1-\frac{1}{n}>1-\varepsilon.$$

i.e. Any number strictly less than 1 is not an upper band for A.

⚠ sup A need not be an element of A ⚠

Fact: If $\sup A\notin A$, then there exists a sequence of element in A that converges to $\sup A$.

(Proof of this fact is an exercise, see HW4)

Claim 1 If $A\subseteq B$, and $\sup A$ & $\sup B$ exist, then $\sup A\le\sup B$

Proof Let $s=\sup B$. Since s is an upper bound for B and $A\subseteq B$ (every element in A is also an element of B) it follows that s is also an upper bound for A.

Since $\sup A$ is the least upper bound for A, $\sup A\le s$. □

Claim 2 If A is non-empty & $m\le a\le M$ $\forall a\in A$, then $m\le\sup A\le M$

Proof: Note that the Axiom of Completeness guarentees $\sup A$ exists.

Since $\sup A$ is the least upper band for A & M is an upper bound $\Rightarrow \sup A\le M$

Since $\sup A$ is an upper band for A $\Rightarrow a\le\sup A$ $\forall a\in A$,

so the fact that $m\le a$ $\forall a\in A\Rightarrow m\le\sup A$. □

Definition (Greatest Lower-Bound)

A real number i is the greatest lower bound for a set $A \subseteq \mathbb{R}$ if it satisfies the following:

(i)' $i \leq a \ \forall a \in A$ (i is a lower bound for A)

(ii)' $\forall \varepsilon > 0 \ \exists a \in A$ with $a < i + \varepsilon$

(any number less that i is not a lower-bound for A)

* The greatest lower bound is also frequently called the infimum of A and denoted by inf(A).

Claim 3 If A is a non-empty bounded subset of $\mathbb{R}$, then

$$\inf(A) = -\sup(-A)$$

where $-A = \{-a : a \in A\}$.

Proof Let $s = \sup(-A)$, note that AoC ensures this exists.

Since $s \geq -a \ \forall a \in A$ (by (i) above)

$\Rightarrow -s \leq a \ \forall a \in A$ ($-s$ is a lower bound for A)

Let $\varepsilon > 0$. Since $\exists a \in A$ with $s - \varepsilon < -a$ (by (ii) above) it follows that for this element $a \in A$, $a < -s + \varepsilon$.

Thus $-s = \inf(A)$ since (i)' & (ii)' are satisfied □

Claim 4: If A non-empty & bounded, then $\inf(A) \leq \sup(A)$.

Proof: For any $a \in A$, $\inf(A) \overset{(i)'}{\leq} a \overset{(i)}{\leq} \sup(A)$ □