Limit Laws and more examples

Proposition (Limit Laws)

Suppose $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

1. $\lim_{n\to\infty} (a_n + b_n) = A + B$.

2. $\lim_{n\to\infty} (a_n b_n) = AB$.

   [In particular $\lim_{n\to\infty} k a_n = kA$ for any $k \in \mathbb{R}$]

3. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ provided $B \neq 0$.

Applications:

(1) **Claim**: $\lim_{n\to\infty} \frac{5n+1}{3n-2} = \frac{5}{3}$

Proof

Using limit laws 1, 2 (with $k=-2$) and 3 (since $3-2(0)\neq 0$).

$$\frac{5n+1}{3n-2} = \frac{5+\frac{1}{n}}{3-2\frac{1}{n}} \to \frac{5+0}{3-2(0)} = \frac{5}{3}.$$

(2) **Proposition** ("Squeeze Theorem")

If $a_n \leq b_n \leq c_n$ for all sufficiently large $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ too.

Proof: Since $0 \leq b_n - a_n \leq c_n - a_n$ $\forall$ suff. large $n \in \mathbb{N}$ and $\lim_{n\to\infty} (a_n - c_n) = L$ (by limit law 1 & 2 (with $k=-1$)) it follows by "Baby Squeeze" that

$$\lim_{n\to\infty} (b_n - a_n) = 0 \Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} ((b_n - a_n) + a_n) = 0 + L = L. \qquad \square$$

limit law 1

Proof of Limit Laws:

Proof of 1: Let $\varepsilon > 0$. Since $a_n \to A$ we know $\exists N_1$ such that $n > N_1$ implies $|a_n - A| < \varepsilon/2$ (since $\frac{\varepsilon}{2} > 0$)

Similarly, since $b_n \to B$ we know $\exists N_2$ such that $n > N_2 \Rightarrow |b_n - B| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. If $n > N$, then

since $n > N_1$ & $n > N_2$.

$$|(a_n + b_n) - (A+B)| = |(a_n - A) + (b_n - B)|$$
$$\leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Proof of 2: Since $\{a_n\}$ convergent we know $\exists M > 0$ such that $|a_n| \leq M$ $\forall n \in \mathbb{N}$.

Since $|a_n b_n - AB| = |(a_n b_n - a_n B) + (a_n B - AB)|$
$$\leq |a_n||b_n - B| + |B||a_n - A| \leq M|b_n - B| + |B||a_n - A|.$$

and $[M|b_n - B| + |B||a_n - A|] \to 0$ (by limit law 1 above & Ex 2 previously)

it follows by "Baby Squeeze" that $\lim\limits_{n\to\infty} (a_n b_n - AB) = 0 \Leftrightarrow \lim\limits_{n\to\infty} a_n b_n = AB$ $\square$

Proof of 3: Since $\frac{a_n}{b_n} = a_n\left(\frac{1}{b_n}\right)$ it suffices in light of "limit law 2" above

to establish: <u>Claim</u>: If $\lim\limits_{n\to\infty} b_n = B$ with $B \neq 0$, then $\lim\limits_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}$.

Proof of Claim: Since $b_n \to B$ (& $B \neq 0$) we know $\exists N$ such that $n > N \Rightarrow |b_n - B| < \frac{|B|}{2}$ (since $\frac{|B|}{2} > 0$).

It follow that if $n > N$ then $|B| = |(B - b_n) + b_n| \leq |B - b_n| + |b_n| < \frac{|B|}{2} + |b_n|$
& hence that $|b_n| > \frac{|B|}{2}$ (provided $n > N$).

So if $n > N$, then $\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|b_n - B|}{|b_n||B|} < \frac{2}{|B|^2}|b_n - B|$, since

$\frac{2}{|B|^2}|b_n - B| \to 0$ it follows from "Baby Squeeze" that $\lim\limits_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}$. $\square$