Theorem (Order Limit Law)

Let $\{a_n\}$ be a convergent sequence with $\lim_{n\to\infty} a_n = a$.

(i) If $a_n \leq U$ for all $n \in \mathbb{N}$, then $a \leq U$ also

(ii) If $a_n \geq L$ for all $n \in \mathbb{N}$, then $a \geq L$ also.

Proof

We only prove (ii) here.

Suppose for the sake of contradiction that $a < L$.

From the definition of convergence, with $\varepsilon = L - a > 0$, we know that there exists a number N such that

$$n > N \text{ implies } |a_n - a| < L - a.$$

* Since $a_n - a \leq |a_n - a|$ for all $n \in \mathbb{N}$ it follows that

$$a_n - a < L - a \quad \text{for all } n > N$$

and hence that $a_n < L$ for all $n > N$ $\lightning$

* This contradicts the assumption that $a_n \geq L$ for <u>all</u> $n \in \mathbb{N}$.

$\square$