$\boxed{\text{Theorem: } \{a_n\} \text{ convergent} \Rightarrow \{a_n\} \text{ bounded.}}$

Note that by taking the contrapositive we get:

Corollary: If $\{a_n\}$ is not bounded, then $\{a_n\}$ is divergent.

Example: The sequence $a_n = \frac{n^2}{2n+1}$ is divergent since it is not bounded above

[If $\exists M > 0$ such that $\frac{n^2}{2n+1} \leq M \; \forall n \in \mathbb{N}$, then $\frac{n^2}{3n} \leq M \; \forall n \in \mathbb{N}$ too, (since $\frac{n^2}{2n+1} \geq \frac{n^2}{2n+n} = \frac{n^2}{3n}$) & hence $n \leq 3M \; \forall n \in \mathbb{N}$ $\lightning$ But this contradicts the fact that $\mathbb{N}$ is unbounded above.]

Proof of Theorem

Since $\exists\, a \in \mathbb{R}$ such that $\lim_{n \to \infty} a_n = a$ we know $\exists\, N$ such that

$$n > N \text{ implies } |a_n - a| < 1 \qquad (\text{take } \varepsilon = 1 \text{ in defn})$$

Since $|a_n| = |(a_n - a) + a| \underset{\uparrow \, \Delta\text{-inequality}}{\leq} |a_n - a| + |a|$ it follows that

$$|a_n| \leq 1 + |a| \quad \text{for all } n > N.$$

and hence that $|a_n| \leq \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |a|\}$. $\square$

⚠ Converse of this Theorem is FALSE ⚠

Ex: $(-1)^n$ is bounded, but not convergent.