Theorem (Limits are unique)

Suppose $\lim_{n\to\infty} a_n = a$ & $\lim_{n\to\infty} a_n = b$, then $a = b$.

Proof

We will suppose $a \neq b$ and seek a contradiction.

We may further assume that $a < b$ (otherwise swith them).

Set $\varepsilon := \frac{b-a}{2} > 0$.

- Since $\lim_{n\to\infty} a_n = a$ we know $\exists\, N_1$ such that

  if $n > N_1$ we have $|a_n - a| < \frac{b-a}{2}$

- Since $\lim_{n\to\infty} a_n = b$ we know $\exists\, N_2$ such that

  if $n > N_2$ we have $|a_n - b| < \frac{b-a}{2}$

Thus is $n > \max\{N_1, N_2\}$ it follows that

$$
\begin{aligned}
b-a = |b-a| &= |(b-a_n)+(a_n-a)| \\
&\overset{\Delta\text{-inequality}}{\leq} |b-a_n| + |a_n-a| \\
&< \underbrace{\frac{b-a}{2}}_{\text{since } n>N_2} + \underbrace{\frac{b-a}{2}}_{\text{since } n>N_1} = b-a
\end{aligned}
$$

* We have just shown that $b-a < b-a$ ↯ a contradiction!

$\square$