Claim The sequence $a_n = (-1)^n$ diverges

Proof (Contradiction)

We suppose that $\lim_{n\to\infty} (-1)^n$ exists and equals some $a \in \mathbb{R}$.

It follows that for any $\varepsilon > 0$ there must exist a number N such that $n > N$ implies $|(-1)^n - a| < \varepsilon$, so in particular (with $\varepsilon = 1$) there must exist N such that $n > N$ implies

$$|(-1)^n - a| < 1.$$

Notice that if $n > N$ and even, then $|1 - a| < 1$ ①

while if $n > N$ and odd, then $|-1 - a| < 1$ ②

(*) Inequality ① tells us that $a \in (0,2)$

while inequality ② tell us that $a \in (-2,0)$

↯ This is a contradiction since $(-2,0)$ & $(0,2)$ are disjoint. □

(a cannot live in both $(-2,0)$ and $(0,2)$).