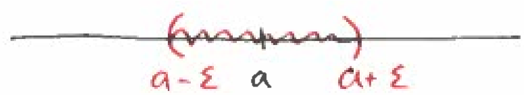
Convergence of Sequences

Let $a \in \mathbb{R}$ and $\varepsilon > 0$.

The ε-neighborhood of a is the open interval $(a-\varepsilon, a+\varepsilon)$,



it consists of all real numbers x such that $|x-a| < \varepsilon$.

"Informal definition of convergence"

We say that a sequence $\{a_n\}$ converges to the real number a if for any $\varepsilon > 0$, the terms of the sequence $\{a_n\}$ "eventually" belong to the ε-neighborhood of a.

¿ But what does "eventually" mean? Lets make this precise:

⊛ **Definition** (Convergence of a Sequence)

- We say that a sequence $\{a_n\}$ converges to a, and write $\lim_{n\to\infty} a_n = a$ (or $a_n \to a$), if:

  > For every $\varepsilon > 0$, there exists a number N such that if $n > N$, then $|a_n - a| < \varepsilon$.

- If there is no $a \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = a$ we say that the sequence $\{a_n\}$ diverges.

Remarks (on equivalent statements)

1. $\lim_{n\to\infty} a_n = a \iff \lim_{n\to\infty} (a_n - a) = 0$.

2. $\lim_{n\to\infty} a_n = a \iff$ For every $\varepsilon > 0$, only finitely many terms of $\{a_n\}$ live outside $(a-\varepsilon, a+\varepsilon)$.

Examples

(1) Claim: $\lim_{n\to\infty} \frac{1}{n} = 0$.

[Rough Work / Discussion prior to formal proof:

Given any $\varepsilon > 0$, we want to produce N so that $n > N$ ensures that $\underbrace{\left|\frac{1}{n} - 0\right|}< \varepsilon$.

→ We focus on the LHS and simplify as follows

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n}$$

So we are looking for a number N so that if $n > N$ this will ensure that $\frac{1}{n} < \varepsilon$. Taking reciprocal we see that $\frac{1}{n} < \varepsilon \iff n > \varepsilon^{-1}$, we should thus take N to be ε^{-1}.]

Formal Proof: Let $\varepsilon > 0$ & set $N = \varepsilon^{-1}$.

If $n > N$ it follows that

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon \quad \text{since } n > N \text{ ensures } n > \varepsilon^{-1} \iff \frac{1}{n} < \varepsilon. \quad \square$$

② Claim: $\lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2}$

Rough Work

Given any $\varepsilon > 0$ we want to produce a number N so that $n > N$ will ensure that $\left|\frac{n+1}{2n-1} - \frac{1}{2}\right| < \varepsilon$.

Focusing on the LHS and simplifying we see that

$$\left|\frac{n+1}{2n-1} - \frac{1}{2}\right| = \left|\frac{2(n+1)-(2n-1)}{2(2n-1)}\right| = \left|\frac{3}{4n-2}\right| \underset{\uparrow}{=} \frac{3}{4n-2}.$$

since $4n-2>0$ $\forall n \in \mathbb{N}$.

Since $\frac{3}{4n-2} < \varepsilon \Leftrightarrow \frac{4n-2}{3} > \frac{1}{\varepsilon}$

$$\Leftrightarrow 4n > \frac{3}{\varepsilon} + 2$$

$$\Leftrightarrow n > \frac{3}{4\varepsilon} + \frac{1}{2}$$

we see that we should take $N = \frac{3}{4\varepsilon} + \frac{1}{2}$.

Formal Proof: Let $\varepsilon > 0$ & set $N = \frac{3}{4\varepsilon} + \frac{1}{2}$.

If $n > N$ then it follows that

$$\left|\frac{n+1}{2n-1} - \frac{1}{2}\right| = \frac{3}{4n-2} < \varepsilon \text{ since } n > \frac{3}{4\varepsilon} + \frac{1}{2} \Leftrightarrow \frac{4n-2}{3} > \frac{1}{\varepsilon}$$

$$\Leftrightarrow \frac{3}{4n-2} < \varepsilon.$$

$\square$.

③ <u>Claim</u> : $\lim_{n\to\infty} \frac{n^2+2}{5n^2+1} = \frac{1}{5}$

[

<u>Rough Work</u>

We start by simplifying:

since $\frac{9}{25n^2+5} > 0 \ \forall n \in \mathbb{N}$.

$$\left| \frac{n^2+2}{5n^2+1} - \frac{1}{5} \right| = \left| \frac{5(n^2+2)-(5n^2+1)}{25n^2+5} \right| = \frac{9}{25n^2+5}$$

$$\leq \frac{9}{25n^2}.$$

since $25n^2 \leq 25n^2+5$
$\Updownarrow$
$\frac{9}{25n^2} \geq \frac{9}{25n^2+5}$

Since $\frac{9}{25n^2} < \varepsilon \Leftrightarrow n > \frac{3}{5\varepsilon^{1/2}}$

we should take $\underline{N = \frac{3}{5\varepsilon^{1/2}}}$.

]

<u>Formal Proof</u> : Let $\varepsilon > 0$ & set $N = \frac{3}{5\varepsilon^{1/2}}$.

If $n > N$ it follows that

$$\left| \frac{n^2+2}{5n^2+1} - \frac{1}{5} \right| = \frac{9}{25n^2+5} \leq \frac{9}{25n^2} < \varepsilon$$

since $n > \frac{3}{5\varepsilon^{1/2}} \Rightarrow \frac{25n^2}{9} > \frac{1}{\varepsilon} \Rightarrow \frac{9}{25n^2} < \varepsilon$ $\square$

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④ <u>Claim</u> $\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6} = 4$

<u>Rough Work</u>

As always we start by simplifying $|a_n - a|$, in this case:

$$\left|\frac{4n^3+3n}{n^3-6} - 4\right| = \left|\frac{4n^3+3n-4(n^3-6)}{n^3-6}\right|$$

$$= \left|\frac{3n+24}{n^3-6}\right|$$

<u>if</u> $n>1$

$$= \frac{3n+24}{n^3-6}$$

replace 24 with $24n$ to simplify numerator (while making it bigger)

$$\leq \frac{3n+24n}{n^3-6}$$

<u>if</u> $n>2$, it follows that $n^3-6 > n^3/2$

$\Updownarrow$

$\frac{1}{n^3-6} < \frac{1}{n^3/2}$

which simplifies the denominator (while make it smaller)

$$< \frac{27n}{n^3/2} = \frac{54}{n^2}$$

$(*)$ since $\frac{54}{n^2} < \varepsilon$ if $n > \sqrt{\frac{54}{\varepsilon}}$

we should take $N = \max\left\{2, \sqrt{\frac{54}{\varepsilon}}\right\}$

<u>Formal Proof</u>: Let $\varepsilon > 0$ & set $N = \max\left\{2, \sqrt{54/\varepsilon}\right\}$.

If $n>N$ it follows that

$$\left|\frac{4n^3+3n}{n^3-6} - 4\right| \underset{\text{since } n>1}{=} \frac{3n+24}{n^3-6} \underset{\text{since } n>2}{\leq} \frac{27n}{n^3/2} = \frac{54}{n^2} \underset{\text{since } n>\sqrt{\frac{54}{\varepsilon}}}{<} \varepsilon$$

$\square$