Two Examples of showing a sequence is monotone & bounded

Claim 1

The sequence $\{a_n\}$ defined by $a_{n+1} = \dfrac{4a_n+3}{a_n+2}$ $\forall n \in \mathbb{N}$ & $a_1 = 4$

is decreasing and bounded below by 3.

Proof

- Proof that $a_n \geq 3$ $\forall n \in \mathbb{N}$: (by Induction)

  $n=1$: Since $a_1 = 4 \geq 3$ ✓

  Suppose $a_n \geq 3 \Leftrightarrow a_n - 3 \geq 0$ for some given $n \in \mathbb{N}$,

  then $a_{n+1} - 3 = \dfrac{4a_n+3}{a_n+2} - 3 = \dfrac{a_n-3}{a_n+2} = \dfrac{a_n-3}{(a_n-3)+5} \geq 0.$

- Proof that $\{a_n\}$ is decreasing:

$$a_{n+1} - a_n = \frac{4a_n+3}{a_n+2} - a_n = -\frac{(a_n^2-2a_n-3)}{a_n+2} = -\frac{(a_n-3)(a_n+1)}{a_n+2}$$

$$\leq 0 \quad \forall n \in \mathbb{N}$$

[since $a_n - 3 \geq 0$ $\forall n \in \mathbb{N}$]

□

Alternative Approach:

Write $a_{n+1} = \dfrac{4a_n+3}{a_n+2} = \dfrac{4a_n+8-5}{a_n+2} = 4 - \dfrac{5}{a_n+2}.$

- Proof that $a_n \geq 3$ $\forall n$: (Induction)

  $n=1$: Since $a_1 = 4 \geq 3$ ✓

  Again suppose $a_n \geq 3$ for some $n \in \mathbb{N}$, it follows that

$$\frac{5}{a_n+2} \le 1 \iff 4 - \frac{5}{a_n+2} \geq 3 \iff a_{n+1} \geq 3 .$$

- Proof that $\{a_n\}$ decreasing: (Induction)

$n=1$: $a_2 = 4 - \frac{5}{4+2} = \frac{19}{6} \le 4 = a_1$ ✓

Suppose $a_{n+1} \le a_n$ for some $n \in \mathbb{N}$, then

$$\frac{5}{a_{n+1}+2} \geq \frac{5}{a_n+2} \iff 4 - \frac{5}{a_{n+1}+2} \le 4 - \frac{5}{a_n+2}$$

$$\iff a_{n+2} \le a_{n+1} \qquad \square$$

Claim 2

$\{a_n\}$ defined by $a_{n+1} = \dfrac{3a_n+2}{a_n+2}$ $\forall n \in \mathbb{N}$ & $a_1 = 1$ is increasing and bounded above by 2.

Proof

Write $a_{n+1} = \dfrac{3a_n+6-4}{a_n+2} = 3 - \dfrac{4}{a_n+2}$.

Proof that $\{a_n\}$ is increasing: (Induction)

$n=1$: $a_2 = 3 - \frac{4}{3} = \frac{5}{3} \geq 1 = a_1$ ✓

Suppose $a_{n+1} \geq a_n$ for some $n \in \mathbb{N}$, then

$$a_{n+2} = 3 - \frac{4}{a_{n+1}+2} \underset{\substack{\uparrow \\ \text{since } a_{n+1} \geq a_n}}{\geq} 3 - \frac{4}{a_n+2} = a_{n+1} .$$

Proof that $a_n \le 2$ $\forall n \in \mathbb{N}$: (Induction)

$n=1$: $a_1 = 1 \le 2$ ✓

Suppose $a_n \le 2$ for some $n \in \mathbb{N}$, then $a_{n+1} = 3 - \dfrac{4}{a_n+2} \overset{\substack{\text{since } a_n \le 2 \\ \downarrow}}{\le} 3 - \dfrac{4}{2+2} = 2$.

$\square$