Monotonicity

Definition (Increasing & Decreasing)

- A sequence $\{a_n\}$ is increasing if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$, and strictly increasing if $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$.
- A sequence $\{a_n\}$ is decreasing if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$, and strictly decreasing if $a_n > a_{n+1}$ $\forall n \in \mathbb{N}$.
- A sequence which is either increasing or decreasing is called monotone (or monotonic).

Examples

1. $a_n = n^2$ is strictly increasing [& hence also increasing and monotone].
2. $a_n = \frac{1}{n}$ is strictly decreasing
3. $a_n = (-1)^n$ is neither increasing nor decreasing.
4. $a_n = 1$ $\forall n \in \mathbb{N} \iff 1, 1, 1, 1, \ldots$ is both increasing and decreasing
5. $1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, \ldots$ is increasing.

* A useful strategy when trying to show that a sequence is increasing (say) is the fact that

$$a_{n+1} \geq a_n \; \forall n \in \mathbb{N} \iff a_{n+1} - a_n \geq 0 \; \forall n \in \mathbb{N}.$$

Ex $a_n = n + \frac{1}{n}$ is strictly increasing since

$$a_{n+1} - a_n = \left((n+1) + \frac{1}{n+1}\right) - \left(n + \frac{1}{n}\right) = 1 + \frac{1}{n+1} - \frac{1}{n} = \frac{n^2+n+1}{n^2+n} > 0 \quad \forall n \in \mathbb{N}.$$

Proposition

- $\{a_n\}$ is increasing $\iff a_m \geq a_n \; \forall m > n$.
- $\{a_n\}$ is decreasing $\iff a_m \leq a_n \; \forall m > n$.

Proof of 2nd claim (1st claim done in class)

($\Leftarrow$) If $a_m \leq a_n \; \forall m > n$, then in particular ($m = n+1$)

$$a_{n+1} \leq a_n \; \forall n \in \mathbb{N} \iff \{a_n\} \text{ decreasing}.$$

($\Rightarrow$) If $\{a_n\}$ decreasing, then $a_{n+1} \leq a_n \; \forall n \in \mathbb{N}$.

$\Updownarrow$

If $m > n$, write $a_{n+1} - a_n \leq 0 \; \forall n \in \mathbb{N}$

$$a_m - a_n = (a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+2} - a_{n+1}) + (a_{n+1} - a_n)$$

Since each term on the RHS is ≤ 0 it follows that

$$a_m - a_n \leq 0 \iff a_m \leq a_n. \qquad \square$$