Boundedness

Defn (Boundedness)

- A sequence $\{a_n\}$ is called <u>bounded above</u> if $\exists\, U \in \mathbb{R}$ with the property that $a_n \leq U \ \forall n \in \mathbb{N}$.

  [U is called <u>an</u> upper bound for $\{a_n\}$]

- A sequence $\{a_n\}$ is called <u>bounded below</u> if $\exists\, L \in \mathbb{R}$ with the property that $a_n \geq L \ \forall n \in \mathbb{N}$.

  [L is called <u>a</u> lower bound for $\{a_n\}$]

- A sequence which is <u>both</u> bounded above & below is called <u>bounded</u>

Examples

① $a_n = \frac{1}{n}$ is bounded above by 1 & below by 0,
[It is also bounded above by 17 & below by -10]
and hence is bounded.

② $a_n = \dfrac{2n}{n+1}$ is bounded above by 2, since

$$\frac{2n}{n+1} \leq \frac{2n}{n} = 2 \quad \forall n \in \mathbb{N}$$

and below by 1, since $\dfrac{2n}{n+1} \geq \dfrac{2n}{n+n} = 1 \ \forall n \in \mathbb{N}$.

(3) $a_n = n^2+1$ is bounded below by 2, but <u>not</u> bounded above.

<u>Proof that it is not bounded above</u> (by contradiction)

Suppose $\{a_n\}$ were bounded above, i.e. $\exists\, U \in \mathbb{R}$ such that $n^2+1 \leq U \quad \forall n \in \mathbb{N}$, but since this U must be ≥ 2 (why?) it follows that

$$n^2 \leq U-1 \quad \forall n \in \mathbb{N} \Leftrightarrow n \leq \sqrt{U-1} \quad \forall n \in \mathbb{N}$$

$$\Leftrightarrow \mathbb{N} \text{ are bounded! } \lightning$$

(4) $a_n = (-1)^n n$ is neither bounded below or above.

<u>Proposition</u>

If $\{a_n\}$ bounded, then $\exists\, M \geq 0$ such that $|a_n| \leq M \quad \forall n \in \mathbb{N}$

<u>Proof</u>

Since $\{a_n\}$ bounded, $\exists\, L, U \in \mathbb{R}$ such that

$$L \leq a_n \leq U \quad \forall n \in \mathbb{N}$$

Since $U \leq |U|$ & $L \geq -|L|$ (basic properties of $|\cdot|$) it follows that

$$-|L| \leq a_n \leq |U| \quad \forall n \in \mathbb{N}.$$

If we set $M = \max\{|L|, |M|\}$ it follows that

$$-M \leq a_n \leq M \quad \forall n \in \mathbb{N} \Longleftrightarrow |a_n| \leq M \quad \forall n \in \mathbb{N}$$

$\square$