Sequences

- **Definition** A sequence is an ordered list of objects.

In this class, when we speak of a sequence will mean an infinite sequence (unless specified otherwise), and the objects in this list will almost always be real numbers.

- **Notation**: We usually denote the n^{th}-term of a sequence by a_n (or another letter in place of "a").

The entire sequence is written as

$$a_1, a_2, a_3, \dots \quad \text{OR} \quad \{a_n\}_{n=1}^{\infty} \quad (\text{sometimes just } \{a_n\}).$$

- **Examples** (Explicitly defined sequences)

① $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ is $a_n = \frac{n-1}{n}$ for all $n \in \mathbb{N}$

② $a_n = \frac{1}{n^2}\ \forall n \in \mathbb{N} \iff 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots$

"for all"

③ $a_n = \frac{(-1)^{n+1}}{n}\ \forall n \in \mathbb{N} \iff 1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \frac{-1}{6}, \dots$

④ $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, 1, \dots \iff a_n = \begin{cases} 1 & \text{if } n \text{ odd.} \\ \frac{1}{k+1} & \text{if } n = 2k \text{ for some } k \in \mathbb{N}. \end{cases}$

- Examples (Recursively defined sequences)

① $a_{n+1} = a_n + \frac{1}{n+1}$ $\forall n \in \mathbb{N}$ & $a_1 = 1$

$\Leftrightarrow$ $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, \ldots$

$\Leftrightarrow$ $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ $\forall n \in \mathbb{N}$.

② [Fibonacci Sequence]

$a_1 = 1, a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ $\forall n \in \mathbb{N}$

$\Leftrightarrow$ $1, 1, 2, 3, 5, 8, 13, 21, \ldots$

③ [Arithmetic Progression]

If a & d are real numbers, then

$a_1 = a$ & $a_{n+1} = a_n + d$ $\forall n \in \mathbb{N}$

$\Leftrightarrow$ $a, a+d, a+2d, \ldots$

$\Leftrightarrow$ $a_n = a + (n-1)d$ $\forall n \in \mathbb{N}$

is an arithmetic progression starting at a with "step size" d.

④ [Geometric Progression] If a & r are reals, then

$a_1 = a$ & $a_{n+1} = r \cdot a_n$ $\forall n \in \mathbb{N}$

$\Leftrightarrow$ $a, ra, r^2a, r^3a, \ldots$

$\Leftrightarrow$ $a_n = r^{n-1} \cdot a$ $\forall n \in \mathbb{N}$

is a geometric progression with "ratio" r.