Binomial Theorem

Recall that if $n \geq k \geq 0$ are integers, then the Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$ if $n \in \mathbb{N}$ & $0! = 1$.

[$\binom{n}{k}$ is often read aloud as "n choose k" since $\binom{n}{k}$ is the number of ways of choosing k elements from a set of n elements]

Theorem (Binomial Theorem) If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$(1+x)^n \underset{\circledast}{=} \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + x^n .$$

Corollary: If $x \geq 0$ and $n \in \mathbb{N}$, then

(i) $(1+x)^n \geq 1+nx$ & (ii) $(1+x)^n \geq \frac{n(n-1)}{2}x^2$.

Before proving the Theorem we establish the following

Lemma If $n \geq k \geq 1$, then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Proof

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!} = \frac{k\,n! + (n-k+1)\,n!}{k!\,(n+1-k)!}$$

$$= \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k} \qquad \square$$

Proof of Binomial Theorem (By Induction)

Base Case ($n=1$): LHS of ⊛ $= (1+x)^1$

RHS of ⊛ $= 1+x$ ✓

Inductive Hypothesis: Suppose ⊛ holds for some $n \in \mathbb{N}$, then

$$(1+x)^{n+1} = (1+x)(1+x)^n.$$

$$\overset{\text{IH}}{=} (1+x)\sum_{k=0}^{n}\binom{n}{k}x^k$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^k + \sum_{k=0}^{n}\binom{n}{k}x^{k+1}$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^k + \sum_{k=1}^{n+1}\binom{n}{k-1}x^k$$

$$= \underbrace{1}_{k=0 \text{ term in } 1^{st} \text{ sum}} + \sum_{k=1}^{n}\binom{n}{k}x^k + \sum_{k=1}^{n}\binom{n}{k-1}x^k + \underbrace{x^{n+1}}_{k=n+1 \text{ term in } 2^{nd} \text{ sum}}$$

$$= 1 + \sum_{k=1}^{n}\left(\binom{n}{k}+\binom{n}{k-1}\right)x^k + x^{n+1}$$

Lemma

$$= 1 + \sum_{k=1}^{n}\binom{n+1}{k}x^k + x^{n+1}$$

since $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$,

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}x^k$$

□