# Three Examples of Proof by Induction

**Claim 1** (The sum of first $n$ odd numbers equals the $n^{th}$ squar

$$1+3+5+\cdots+(2n-1) = n^2 \quad \text{for all } n \in \mathbb{N}. \qquad (\circledast)$$

**Proof**

**Base Case** $(n=1)$ : Since LHS $=1$ & RHS $=1^2$ ✓

**Inductive Hypothesis** : Suppose $\circledast$ holds for some given $n \in \mathbb{N}$.

**Inductive Step** : We now want to show that $1+3+\cdots+(2n-1)+(2n+1)=($

$$1+3+\cdots+(2n-1)+\underbrace{(2n+1)}_{2(n+1)-1} \underset{\text{by I.H.}}{=} n^2+(2n+1) = (n+1)^2. \qquad \square$$

**Claim 2** (Generalized Triangle Inequality)

For any $n \geq 2$ & $x_1, \ldots, x_n \in \mathbb{R}$ we have

$$|x_1+\cdots+x_n| \leq |x_1|+\cdots+|x_n|. \qquad (\circledast)$$

**Proof**

**Base Case** $(n=2)$ : ✓ (This is the triangle inequality)

**Inductive Hypothesis** : Suppose inequality $\circledast$ holds for some $n \in$

It follows that

$$|x_1+\cdots+x_n+x_{n+1}| = |(x_1+\cdots+x_n)+x_{n+1}| \overset{n=2}{\leq} |x_1+\cdots+x_n|+|x_{n+1}|$$

$$\overset{\text{IH}}{\leq} |x_1|+\cdots+|x_n| + |x_{n+1}|$$

## Claim 3

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} \underset{\circledast}{\leq} 2-\frac{1}{n} \quad \text{for all } n\in\mathbb{N}.$$

## Proof

Base Case $(n=1)$: LHS $= 1$ & RHS $= 2-\frac{1}{1} = 1$ ✓

Inductive Hypothesis: Suppose inequality $\circledast$ hold for some $n\in\mathbb{N}$

It then follows that

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}+\frac{1}{(n+1)^2} \overset{\text{IH}}{\leq} \left(2-\frac{1}{n}\right)+\frac{1}{(n+1)^2}$$

$$= 2-\left(\frac{1}{n}-\frac{1}{(n+1)^2}\right).$$

The result will follow if we can show that $\frac{1}{n}-\frac{1}{(n+1)^2} \geq \frac{1}{n+1}$

(do you agree?)

Subclaim: For any $n\in\mathbb{N}$ we have $\frac{1}{n+1} \leq \frac{1}{n}-\frac{1}{(n+1)^2}$.

Proof

$$\frac{1}{n}-\frac{1}{(n+1)^2} = \frac{(n+1)^2-n}{n(n+1)^2} = \frac{n^2+n+1}{n(n+1)^2} \geq \frac{n^2+n}{n(n+1)^2}$$

$$= \frac{1}{n+1}. \qquad \square$$