Absolute Value and Inequalities

Properties of Inequalities

If $x, y, z \in \mathbb{R}$ then the following are true:

a) $x < y$ & $y < z \Rightarrow x < z$

b) $x < y \Rightarrow x + z < y + z$

c) $x < y$ & $z > 0 \Rightarrow xz < yz$

d) $x < y$ & $z < 0 \Rightarrow xz > yz$

e) $0 < x < y \Rightarrow \frac{1}{x} > \frac{1}{y} > 0$.

It is also useful to remember that if $x \in \mathbb{R}$, then exactly one of the following must be true:

(i) $x < 0$ (ii) $x = 0$ (iii) $x > 0$.

Recall that for all $x \in \mathbb{R}$, its absolute value is defined to be its "distance from 0", namely

$$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Note that $|x| \geq 0$ always & $|x| = 0 \Leftrightarrow x = 0$.

Properties of Absolute Value: If $x, y \in \mathbb{R}$, then

(a) $-|x| \leq x \leq |x|$

(b) $|x| \leq y \Leftrightarrow -y \leq x \leq y$

(c) $|xy| = |x||y|$

(d) $|x+y| \leq |x| + |y|$ (Triangle Inequality)

Proof of the "Triangle Inequality"

Let $x, y \in \mathbb{R}$. It follows that

$$|x+y|^2 = (x+y)^2 = x^2 + 2xy + y^2$$
$$= |x|^2 + 2xy + |y|^2$$

using (a) & (e)

$$\leq |x|^2 + 2|x||y| + |y|^2$$
$$= (|x| + |y|)^2$$

and hence that $|x+y| \leq |x| + |y|$ as required $\square$

¿Why? Claim: If $a, b \geq 0$, then $a \leq b \Rightarrow \sqrt{a} \leq \sqrt{b}$.

Proof (Contrapositive)

Suppose $\sqrt{a} > \sqrt{b} \Rightarrow \sqrt{a} - \sqrt{b} > 0$ & $\sqrt{a} + \sqrt{b} > 0$

$$\Rightarrow (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) > 0$$
$$\Rightarrow a - b > 0 \Leftrightarrow a > b. \quad \square$$

* Using Inequalities to prove equalities

- To show that $y = z$ is equivalent to showing $y - z = 0$.
- One way to show $x = 0$ is to show that $x \leq 0$ & $x \geq 0$.
- Another way: Lemma: If $|x| < \varepsilon$ for all $\varepsilon > 0$, then $x = 0$.

Proof of Lemma (Contrapositive) Suppose $x \neq 0$. It then follows that $|x| > 0$, taking $\varepsilon = |x|$ gives a choice of $\varepsilon > 0$ for which $|x|$ is not $< \varepsilon$. $\square$