# Number Systems

cannot always subtract (or divide)

Natural Numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

cannot always divide.

Integers $\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$

cannot always take roots (see below)

Rationals $\mathbb{Q} = \{ \frac{a}{b} : b \in \mathbb{N} \ \&\ a \in \mathbb{Z} \}$

Reals $\mathbb{R} = ?$ Perfect?

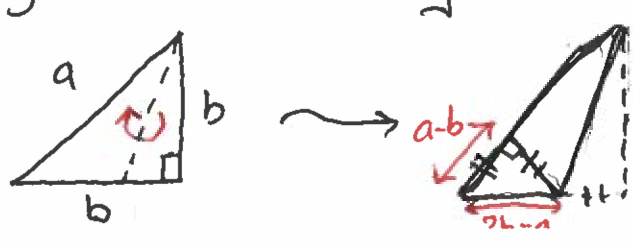
look up what this means!

- $\mathbb{Q}$ and $\mathbb{R}$ are both ordered fields, but $\mathbb{Q} \subset \mathbb{R}$ and in fact $\mathbb{R}$ is a much larger and "complete" collection of numbers. [* Numbers in $\mathbb{R} \setminus \mathbb{Q}$ are called irrational]

Theorem: There is no rational number whose square equals 2, i.e. "$\sqrt{2} \notin \mathbb{Q}$".

Proof

If $\sqrt{2} \in \mathbb{Q}$, there must exist a smallest isosceles right triangle with integer sides. But, given any such triangle one can always construct a smaller one:

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## Finer Properties of $\mathbb{R}$ (no proofs)

- **Archimedean Property (AP)** "there exists"

Given any $\varepsilon > 0$ $\exists$ $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Proof This is nothing but a restatement of the fact that one can find arbitrarily large natural numbers since for any given $\varepsilon > 0$ we can find $n \in \mathbb{N}$ such that $n > \varepsilon^{-1}$ $\square$

$\Updownarrow$

$\frac{1}{n} < \varepsilon$.

Here are two surprising consequences of AP:

**Theorem 1** (Denseness of $\mathbb{Q}$ in $\mathbb{R}$)

If $x, y \in \mathbb{R}$ with $x < y$, then $\exists$ $q \in \mathbb{Q}$ such that $x < q < y$.

**Theorem 2** (Denseness of Irrationals in $\mathbb{R}$)

If $x, y \in \mathbb{R}$ with $x < y$, then $\exists$ $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$.

* We delay a discussion of why AP implies Theorems 1 & 2.

- **Existence of Roots** We delay the proof of this fact also.

**Theorem 3**: For every $x \in \mathbb{R}$ with $x > 0$ and $n \in \mathbb{N}$ $\exists$ unique $y > 0$ such that $y^n = x$ (& $y$ is written $\sqrt[n]{x}$ or $x^{1/n}$)

* The case $n=2$ ensures the square root of any positive real exists.