Math 3100 Assignment 3

Convergence of Sequences

Due at 1:00 pm on Wednesday the 5th of September 2018

1. What happens if we interchange or reverse the order of the quantifiers in the definition of convergence of a sequence?

   (a) *Definition:* A sequence $\{a_n\}$ *verconges* to a if *there exists* an $\varepsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n > N$ implies $|a_n - a| < \varepsilon$.
   Give an example of a vercongent sequence. Can you give an example a vercongent sequence that is divergent? What exactly is being described in this strange definition?

   (b) *Definition:* A sequence $\{a_n\}$ *concongeses* to a if *there exists* a number N such that $n > N$ implies $|a_n - a| < \varepsilon$ *for all* $\varepsilon > 0$.
   Give an example of a concongent sequence. Can you give an example a concongent sequence that is divergent? What exactly is being described in this strange definition?

2. Verify the following <u>using the definition</u> of convergence of a sequence:

   (a) If $a_n \to a$, then $|a_n| \to |a|$. Is the converse true?

   (b) If $\lim\limits_{n\to\infty} b_n = 2$, then $\lim\limits_{n\to\infty} \dfrac{1}{b_n^2} = \dfrac{1}{4}$.
   Hint: First argue that there exists a number N such that if $n > N$, then $1 \le b_n \le 3$.

   (c) If $\{x_n\}$ is bounded (but not necessarily convergent) and $\lim\limits_{n\to\infty} y_n = 0$, then $\lim\limits_{n\to\infty} x_n y_n = 0$.

3. Let $\{a_n\}$ be a convergent sequence with $\lim\limits_{n\to\infty} a_n = a$. Prove the following two statements:

   (a) If $a_n \le b$ for all $n \in \mathbb{N}$, then $a \le b$.

   (b) If $\{a_n\}$ is decreasing, then $a_n \ge a$ for all $n \in \mathbb{N}$.

4. We say that $\{a_n\}$ *diverges to infinity*, and write $\lim\limits_{n\to\infty} a_n = \infty$, if for every $M > 0$ there exists a number N such that $n > N$ implies that $a_n > M$.

   (a) Prove, using the definition above, that if $\lim\limits_{n\to\infty} a_n = \infty$, then $\lim\limits_{n\to\infty} a_n^p = \infty$ for all $p > 0$.

   (b) Suppose $\lim\limits_{n\to\infty} a_n = \infty$.

      i. Prove that if $\{b_n\}$ is a bounded sequence, then $\lim\limits_{n\to\infty} (a_n + b_n) = \infty$.

      ii. Prove that if $\lim\limits_{n\to\infty} b_n = 2$, then $\lim\limits_{n\to\infty} (a_n b_n) = \infty$.

5. Let $y_1 = 2$ and $y_{n+1} = \dfrac{1}{3 - y_n}$ for all $n \in \mathbb{N}$.

   (a) Show that $\{y_n\}$ is decreasing and satisfies $\dfrac{3-\sqrt{5}}{2} \le y_n \le 2$ for all $n \in \mathbb{N}$.

   (b) Conclude that <u>**if**</u> the sequence $\{y_n\}$ converges, then it must converge to $\dfrac{3-\sqrt{5}}{2}$.

   We shall soon establish in class, using the "completeness of the real numbers" (the defining property that distinguishes the reals from the rationals), that bounded monotone sequences of real numbers <u>always</u> converge.