Math 3100 Exam 1

Study Guide and Practice Questions

1. Let $\{x_n\}$ denote a sequence of reals and A denote a subset of $\mathbb{R}$.
   Carefully state the definition of the following:
   (a) A is bounded
   (b) $\sup(A)$ and $\inf(A)$
   (c) $\{x_n\}$ is bounded
   (d) $\{x_n\}$ converges to some real number x
   (e) $\{x_n\}$ diverges to infinity
   (f) a subsequence of $\{x_n\}$
   (g) $\{x_n\}$ is Cauchy

2. Use the definition of convergence to prove the following:
   $$\text{(a)} \quad \lim_{n\to\infty} \frac{3n+5}{n+3} = 3 \qquad\qquad \text{(b)} \quad \lim_{n\to\infty} \frac{2n+1}{2-n} = -2$$

3. Determine which of the following sequences converge and which diverge. Find the value of the limit for those which converge. *Be sure to indicate any limit laws, theorems, or special limits used.*
   $$\text{(a)} \quad a_n = (-1)^n n \qquad \text{(b)} \quad b_n = \frac{2^n}{n^2+2^n} \qquad \text{(c)} \quad c_n = (-1)^n - \frac{1}{n} \qquad \text{(d)} \quad d_n = \sqrt{n+1} - \sqrt{n}$$

4. (a) Let $\{x_n\}$ be defined recursively by $x_1 = 2$ and $x_{n+1} = \dfrac{1+x_n}{2}$ for each $n \in \mathbb{N}$. Show that $\{x_n\}$ converges and find its limit.
   (b) Show that if $x_n = \dfrac{1}{1+n} + \dfrac{1}{2+n} + \cdots + \dfrac{1}{n+n}$ for each $n \in \mathbb{N}$, then $\{x_n\}$ converges.

5. Assume that $\lim_{n\to\infty} a_n = \infty$, that is the sequence $\{a_n\}$ diverges to infinity, and that $\lim_{n\to\infty} b_n = L$ with $L > 0$. Prove the following two statements;
   (a) There exists some $N \in \mathbb{N}$ such that if $n > N$, then $b_n > L/2$.
   (b) The sequence $\{a_n b_n\}$ diverges to infinity.

6. Prove that if $|a_n| \le b_n$ and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

7. Let A be non-empty bounded subset of $\mathbb{R}$. Prove that $\inf(A) = -\sup(-A)$, where $-A = \{-a : a \in A\}$.

8. Carefully state the following:
   (a) The Axiom of Completeness (AoC)
   (b) The Monotone Convergence Theorem (MCT)
   (c) The Bolzano-Weierstrass Theorem (BW)
   (d) The Cauchy Criterion (CC)

9. Prove the following implications:
   (a) $\{x_n\}$ convergent $\implies$ $\{x_n\}$ is bounded
   (b) $\{x_n\}$ convergent $\implies$ $\{x_n\}$ is Cauchy
   (c) AoC $\implies$ MCT
   (d) BW $\implies$ CC (you should also show that Cauchy $\implies$ Bounded)

10. (a) Let $\{x_n\}$ denote a bounded sequence of real numbers.
      i. Give the definition of $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$
      ii. Prove that if $\limsup_{n\to\infty} |x_n| = 0$, then $\lim_{n\to\infty} x_n$ exists and equals 0.
      iii. Prove that if $\{x_n\}$ diverges then it has at least two subsequences that converge to different limits.

    (b) Let $\{b_n\}$ be a sequence of real numbers. Prove that

$$\limsup_{n\to\infty} \frac{1}{1+b_n^2} \leq 1.$$

Indicating why $\limsup_{n\to\infty} x_n$ *exists when* $x_n = \dfrac{1}{1+b_n^2}$ *before proving that it cannot exceed* 1.