Discrete Time Models (2d)

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Systems of Discrete-Time Equations

Systems of Discrete time equations:

$$
\begin{aligned}
R_{n+1} &= a_R R_n + p_R J_n \\
J_{n+1} &= a_J J_n + p_J R_n
\end{aligned}
$$

R_n=Romeo's love/hate for Juliet on day n.
Love if $R_n > 0$, hate when $R_n < 0$, neutral at $R_n = 0$.
No mood swings:

$$a_R, a_J > 0$$

Initial feeling becomes neutral as time progresses:

$$0 < a_R, a_J < 1$$

Love (from partner) could induce love, or hate: p_R, p_J can take positive or negative values. Long term behavior? Stability?

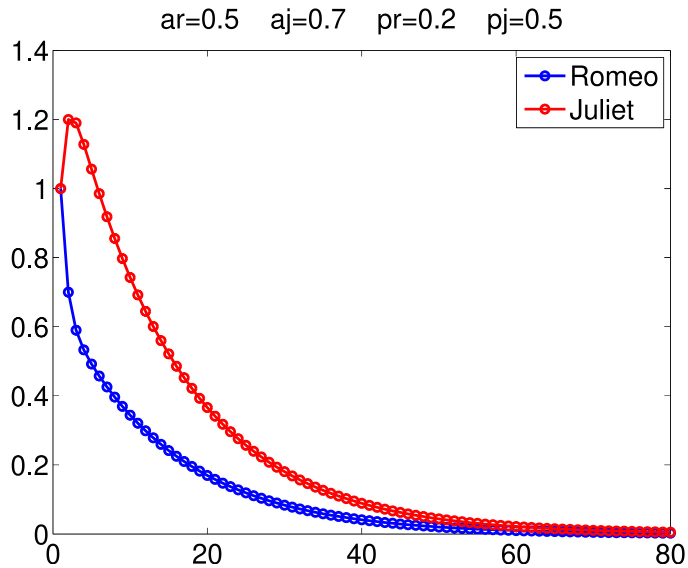
Example 1

$$
\begin{aligned}
R_{n+1} &= a_R R_n + p_R J_n \\
J_{n+1} &= a_J J_n + p_J R_n
\end{aligned}
$$

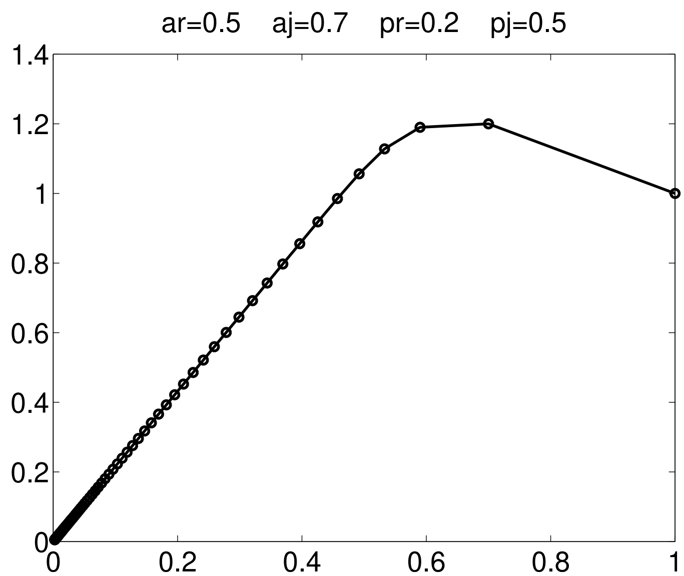
$$a_R = 0.5 \quad a_J = 0.7 \quad p_R = 0.2 \quad p_J = 0.5$$

Anticipated long term behavior?

Long term behavior



State space



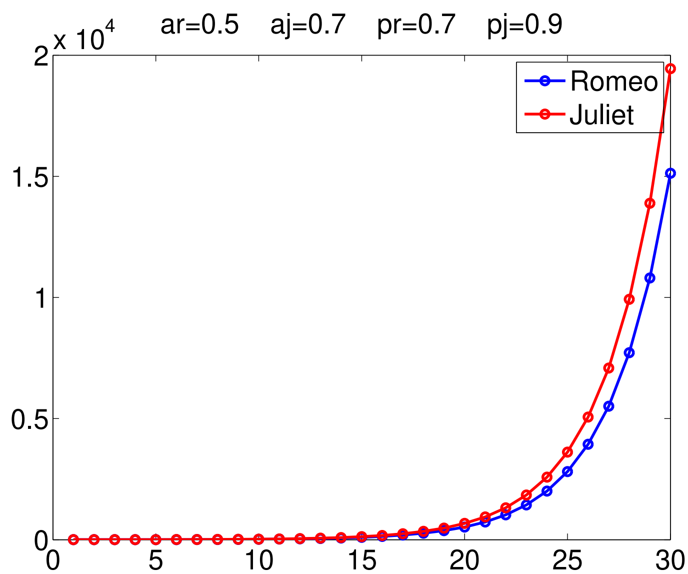
Example 2

$$\begin{aligned} R_{n+1} &= a_R R_n + p_R J_n \\ J_{n+1} &= a_J J_n + p_J R_n \end{aligned}$$

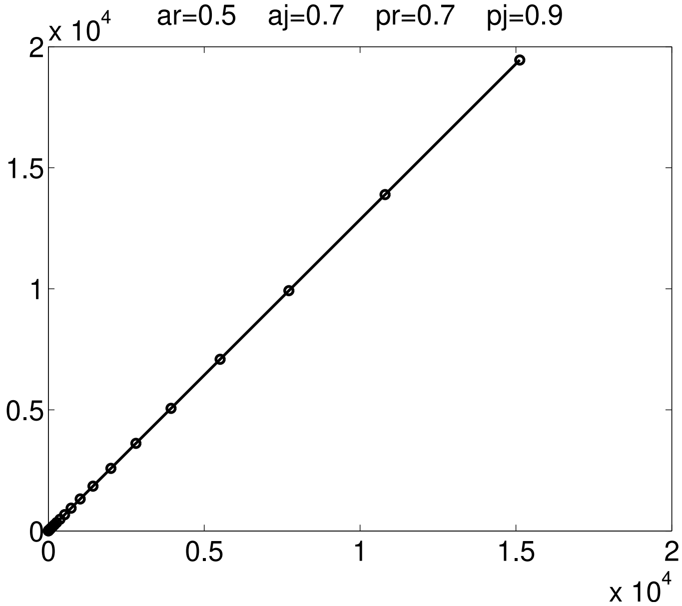
$a_R = 0.5 \quad a_J = 0.7 \quad p_R = 0.7 \quad p_J = 0.9$

Anticipated long term behavior?

Long term behavior



State space



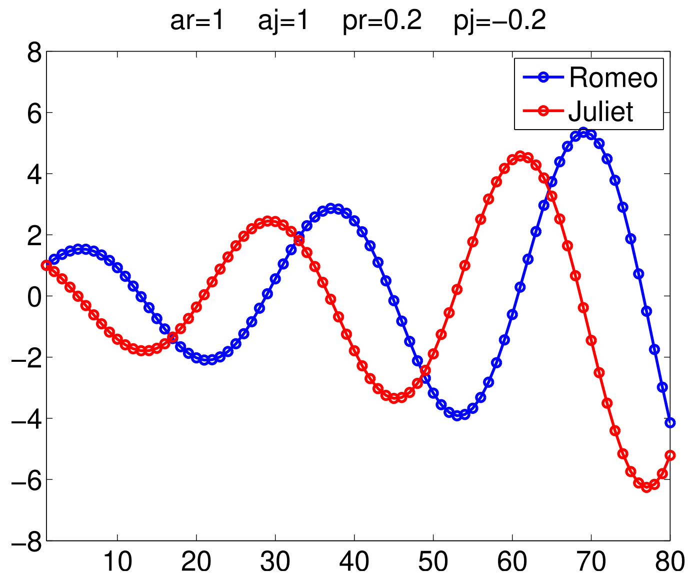
Example 3

$$\begin{aligned} R_{n+1} &= a_R R_n + p_R J_n \\ J_{n+1} &= a_J J_n + p_J R_n \end{aligned}$$

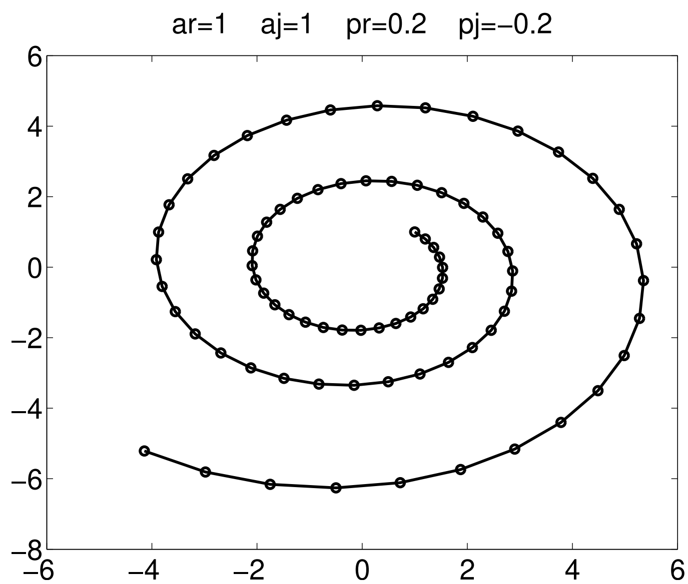
$a_R = 1 \quad a_J = 1 \quad p_R = 0.2 \quad p_J = -0.2$

Anticipated long term behavior?

Long term behavior



State space



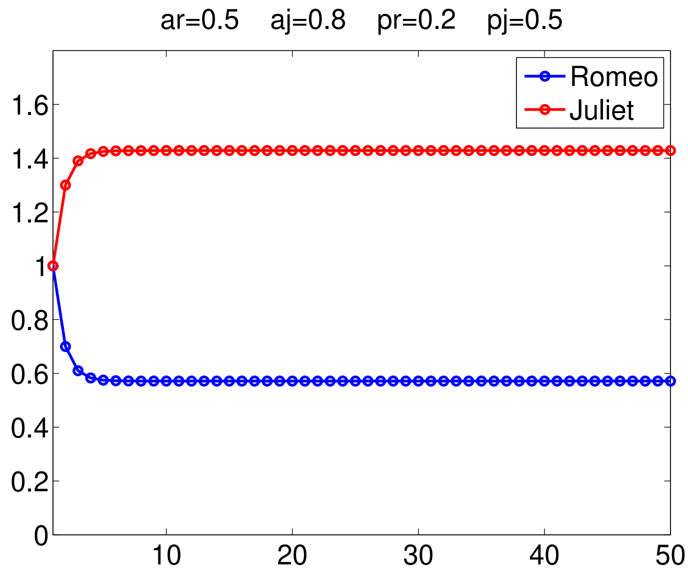
Example 4

$$R_{n+1} = a_R R_n + p_R J_n$$
$$J_{n+1} = a_J J_n + p_J R_n$$

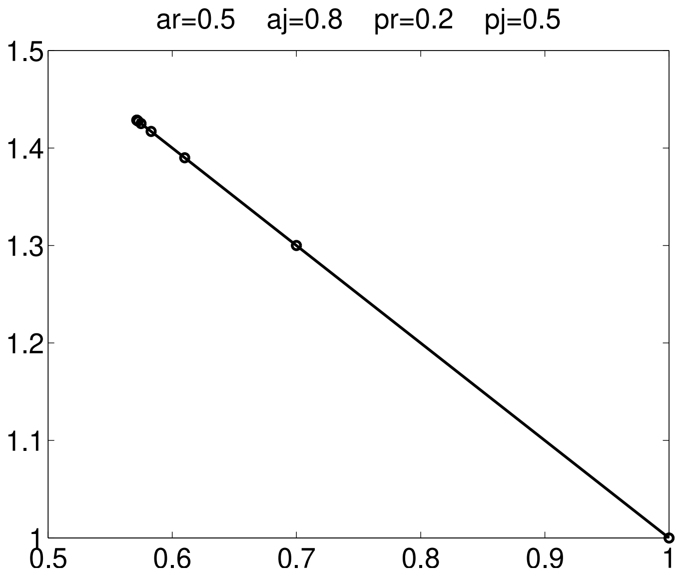
$a_R = 0.5 \quad a_J = 0.8 \quad p_R = 0.2 \quad p_J = 0.5$

Anticipated long term behavior?

Long term behavior



State space



Stability analysis (1d)

$$x_{t+1} = f(x_t)$$

x^* is a fixed point if $x^* = f(x^*)$. Let

$$x_t = x^* + y_t$$

Then

$$\begin{aligned} x_{t+1} &= f(x_t) \\ x^* + y_t &= f(x^* + y_t) \\ x^* + y_{t+1} &= f(x^*) + y_t\, f'(x^*) + \mathcal{O}(|y_t|^2) \\ y_{t+1} &\approx y_t\, f'(x^*) = y_t \end{aligned}$$

x^* is a stable fixed point if $|f'(x^*)| < 1$.

Stability analysis (2d)

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned}$$

Consider a small perturbation from the fixed point (x^*, y^*):

$$\begin{aligned} x_n &= x^* + u_n \\ y_n &= y^* + v_n \end{aligned}$$

Combining above equations, we get:

$$\begin{aligned} x^* + u_{n+1} &= f(x^* + u_n, y^* + v_n) \\ y^* + v_{n+1} &= g(x^* + u_n, y^* + v_n) \end{aligned}$$

Taylor series expansion

For a scalar function f of a single variable x:

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \cdots$$

For a scalar function f of multiple variables $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$:

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \mathbf{h}^T \nabla f(\mathbf{x}) + \frac{1}{2}\mathbf{h}^T \nabla^2 f(\mathbf{x}) \mathbf{h} + \cdots$$

Same equation using only scalar variables:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} + $$

$$+ \frac{1}{2} \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \cdots$$

Taylor series expansion

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} +$$
$$+ \frac{1}{2} \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \cdots$$

After matrix product, we get:

$$f(x_1 + h_1, x_2 + h_2) = f + h_1 f_{x_1} + h_2 f_{x_2} + \tfrac{1}{2} h_1^2 f_{x_1 x_1} +$$
$$+ \tfrac{1}{2} h_2^2 f_{x_2 x_2} + h_1 h_2 f_{x_1 x_2} + \cdots$$

which is similar to the single variable equation:

$$f(x + h) = f + h f' + \tfrac{1}{2} h^2 f'' + \cdots$$

Stability Analysis

$$\begin{aligned} x^* + u_{n+1} &= f(x^* + u_n, y^* + v_n) \\ y^* + v_{n+1} &= g(x^* + u_n, y^* + v_n) \end{aligned}$$

$$\begin{aligned} x^* + u_{n+1} &= f(x^*, y^*) + u_n f_x(x^*, y^*) + v_n f_y(x^*, y^*) \\ y^* + v_{n+1} &= g(x^*, y^*) + u_n g_x(x^*, y^*) + v_n g_y(x^*, y^*) \end{aligned}$$

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}$$

So this is how the difference between the solution and the fixed point changes (grows or shrinks).
How can we use this equation to test stability?

Eigenvalues

Theorem 2.2: The fixed points (x^*, y^*) are stable only if the norm of (both) eigen-values of

$$J = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix}$$

are less than 1. Is there an easier way to check stability?

$$\lambda^2 - \mathrm{tr} J\, \lambda + \det J = 0$$

"Jury conditions"

$$|\mathrm{tr} J| < 1 + \det J < 2$$

Convergence region