1. Consider the following population density model:

$$N_{t+1} = 2 - (N_t - 1)^2$$

(a) (7 pts.) Find the non-negative fixed point(s) and analyze their stability.

$$N^* = 2 - (N^* - 1)^2$$

$$N^* = 2 - (N^{*2} - 2N^* + 1)$$

$$N^{*2} - N^* - 1 = 0$$

$$N^*_{1,2} = \frac{1 \pm \sqrt{1^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$N^*_1 = \frac{1-\sqrt{5}}{2} < 0 \qquad \text{negative}$$

$$N^*_2 = \frac{1+\sqrt{5}}{2} > 0 \qquad \checkmark$$

$$f(N) = 2 - (N-1)^2 \Rightarrow f'(N) = -2(N-1)$$

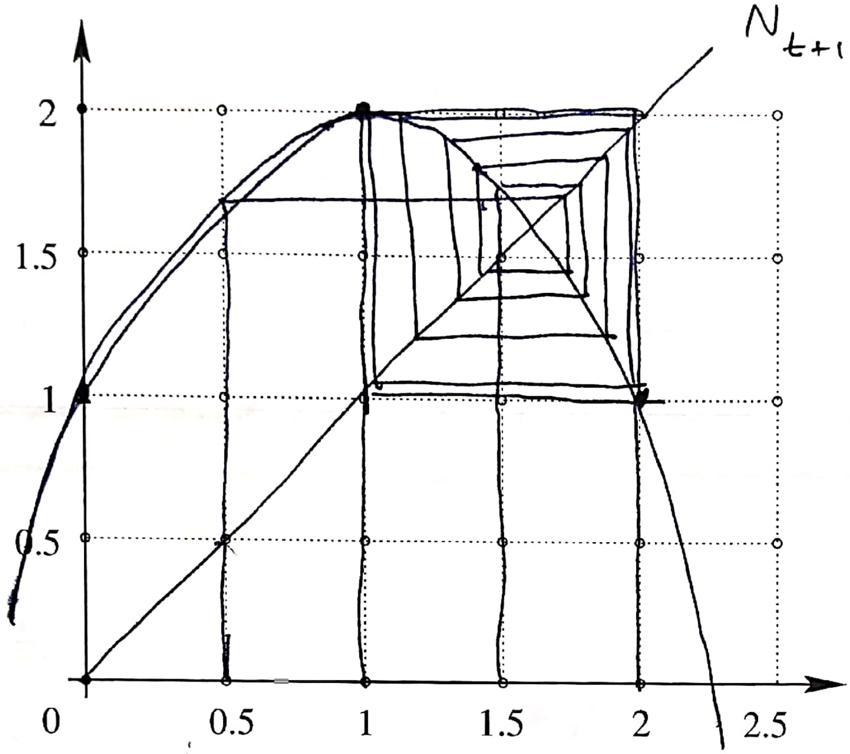
$$\underbrace{\left| -2 \underbrace{\left(\frac{1+\sqrt{5}}{2} - 1\right)}_{= \frac{-1+\sqrt{5}}{2}} \right|}_{\sqrt{5}-1} \overset{?}{<} 1$$

$$|\sqrt{5} - 1| = \sqrt{5} - 1 > 1$$

Then $N^*_x = \dfrac{1+\sqrt{5}}{2}$ is unstable.

(b) (10 pts.) Carefully draw the cobwebs for the following five initial conditions: $N_0 = 0.5,\ 1,\ 1.5,\ 2.$ Do enough iterations to estimate the long term behavior for each solution. Describe and compare the long term behaviors of these four solutions.

$N_{t+1} = 2 - (N_t - 1)^2$



$f(N) = 2 - (N-1)^2$ This is the same figure as $-N^2$; but shifted one unit right, and two units up.

$f(0) = 1,\ f(1) = 2,\ f(2) = 1$

For $N_0 = 1$, $N_1 = N_3 = N_5 = \cdots = 2$ and $N_2 = N_4 = N_6 = \cdots = 1$

For $N_0 = 2$, $N_1 = N_3 = N_5 = \cdots = 1$ and $N_2 = N_4 = N_6 = \cdots = 2$

For $N_0 = 0.5$ and $N_0 = 1.5$; N_t appears to converge to the same two-cycle, $1 \rightleftarrows 2$; whereas $N_0 = 1$ & $N_0 = 2$ already start on the same 2-cycle.

2. (10 pts.) Find all fixed points of the following discrete predator-prey model:

$$x_{n+1} = 4x_n(1 - x_n) - 4x_n y_n$$
$$y_{n+1} = 4x_n y_n - y_n$$

$$x^* = 4x^*(1 - x^*) - 4x^*y^* = 4x^*(1 - x^* - y^*)$$
$$y^* = 4x^*y^* - y^* = y^*(4x^* - 1)$$

$$0 = 4x^*\left(\tfrac{3}{4} - x^* - y^*\right)$$
$$0 = 2\underbrace{y^*}_{y^*=0}\underbrace{(2x^* - 1)}_{x^* = 1/2}$$

Case 1

$$y^* = 0 \Rightarrow 0 = \underbrace{4x^*}_{x^*=0}\underbrace{\left(\tfrac{3}{4} - x^*\right)}_{x^* = 3/4}$$

$(0, 0)$ $(3/4, 0)$

Case 2

$$x^* = 1/2 \Rightarrow 4\tfrac{1}{2}\underbrace{\left(\tfrac{3}{4} - \tfrac{1}{2} - y^*\right)}_{y^* = \frac{1}{4}} = 0$$

$(1/2, 1/4)$

Three fixed points: $(0, 0)$
$(3/4, 0)$
$(1/2, 1/4)$

3. Consider the following differential equation model.

$$u' = u - uv$$
$$v' = 2u - 6v$$

(a) (10 pts.) Find all fixed points, analyze their stability, and classify them (e.g. stable node, center, saddle, etc.) Is coexistence possible?

Fixed points:

$u' = u(1-v) = 0 \Rightarrow$ Either $u=0$ or $v=1$

Case 1. Let $u=0$, then $v' = -6v = 0 \Rightarrow v=0$

$(0,0)$

Case 2. let $v=1$, then $v' = 2u - 6 = 0 \Rightarrow u=3$

$(3,1)$

u- Nullcline(s): $u' = u(1-v) = 0 \Rightarrow$ $u=0$, $v=1$

v- Nullclines(s) = $2u - 6v = 0 \Rightarrow v = \frac{1}{3}u$

$$J(u,v) = \begin{bmatrix} 1-v & -u \\ 2 & -6 \end{bmatrix}$$

$J(0,0) = \begin{bmatrix} 1 & 0 \\ 2 & -6 \end{bmatrix}$ det $= -6 < 0 \Rightarrow$ saddle

$J(3,1) = \begin{bmatrix} 0 & -3 \\ 2 & -6 \end{bmatrix}$ det $= 0-(-6) = 6 > 0 \Rightarrow$ not saddle

tr $= -6 < 0 \Rightarrow$ stable

$(\text{Trace})^2 - 4(\text{Det}) = 36 - 4 \cdot 6 = 36 - 24 = 12 > 0 \Rightarrow$ not saddle.

Stable node

Coexistence is possible; as there is a positive stable fixed point, $(3,1)$.

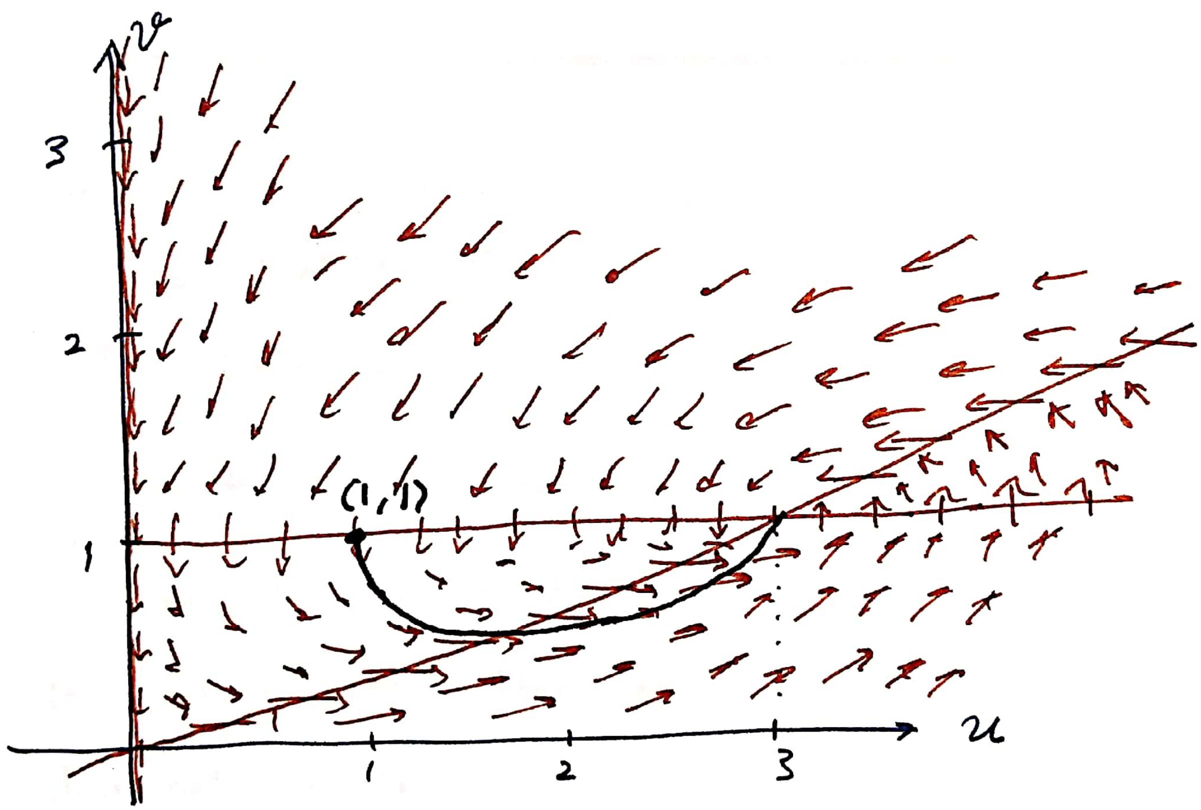
(b) (10 pts.) Do the phase plane analysis. In other words, find and sketch the null-clines, and the phase portrait (direction field). Focus only on the first quadrant ($u \geq 0$, $v \geq 0$). Draw a rough sketch of the solution with initial conditions $u(0) = 1$, $v(0) = 1$.

u-Nullclines : $u' = u - uv = \underbrace{u}_{u=0}\underbrace{(1-v)}_{v=1} = 0$

$\underline{u=0} \quad \underline{v=1}$

v-Nullclines : $v' = 2u - 6v = 2(u - 3v) = 0$

$\underline{v = \frac{1}{3}u}$



For high u & low v : $u' > 0 \quad v' > 0$

For low u & high v : $u' < 0 \quad v' < 0$

4. Fill in the blanks or identify the statement as True or False.

(a) (2 pts.) In problem 2, x represents the predator, and y represents the prey. (True ☐ / False ☒)

(b) (2 pts.) Carrying capacity of the model $x_{n+1} = 2x_n(1 - \frac{x_n}{6})$ is 3. Accepted 6 as well.

(c) (2 pts.) If the model $x_{n+1} = f(x_n)$ has a 2-cycle, such that $a = f(b)$ and $b = f(a)$, then there exists a fixed point c between a and b. (True ☒ / False ☒) Accept both options

(d) (2 pts.) A fixed point of the model $x_{n+1} = f(x_n)$ is also a fixed point of the second iterate map $y_{n+1} = f(f(y_n))$. (True ☒ / False ☐)

(e) (2 pts.) For the model $x_{n+1} = f(x_n)$, if there exists values a and b such that $a = f(b)$ and $b = f(a)$, then both a and b are stable fixed points of the second iterate map $y_{n+1} = f(f(y_n))$. (True ☐ / False ☒)

(f) (2 pts.) Liapunov exponent of the model in the first problem is negative. (True ☒ / False ☐)

(g) (2 pts.) If the Liapunov exponent of a model is positive, then it has no fixed points. (True ☐ / False ☒)

(a) Interaction term $(4xy)$ benefits y.

(b) $x^* = 2x^*(1 - \frac{x^*}{6})$

$0 = 2x^*(\frac{1}{2} - \frac{x^*}{6}) \Rightarrow \underline{x^* = 3}$

(c) f has to be continuous.

(e) It does not have to be stable

(f) It is not chaotic. There exists a stable 2-cycle.

(g) Just not stable.