Discrete logistic MAP: $x_{n+1} = r x_n(1-x_n)$ ①

Second map: $x_{n+1} = r x_{n+1}(1-x_{n+1})$

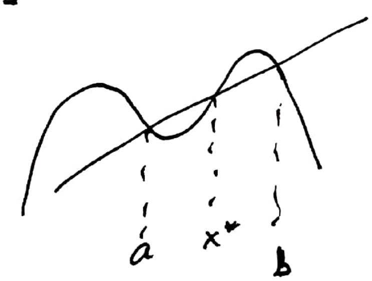
$$= r^2 x_n(1-x_n)(1-r x_n(1-x_n))$$

$$f(x) = \underbrace{r^2 x(1-x)}_{r^2(x-x^2)}\underbrace{(1-rx(1-x))}_{1-rx+rx^2}$$

$$f'(x) = r^2\left[(1-2x)(1-rx(1-x)) + x(1-x)\underbrace{(-r+2rx)}_{-r(1-2x)}\right]$$

$$= r^2(1-2x)\left[1-rx(1-x) - rx(1-x)\right]$$

$$f'(x) = r^2(1-2x)(1-2rx(1-x))$$



$f(a)=a$, $f(b)=b$, $a = rb(1-b)$
$b = ra(1-a)$

$$f'(a) = r^2(1-2a)(1-\underbrace{2ra(1-a)}_{=b}) = r^2(1-2a)(1-2b)$$

We need $f'(a) = -1 = r^2(1-2a)(1-2b)$

$$(1-2a)(1-2b) = -\frac{1}{r^2}$$

$$\cancel{1} - 2(a+b) + 4ab = -\frac{1}{r^2} - 1 \qquad (*)$$

$(I): a = rb(1-b)$

$(II): b = ra(1-a)$

$(I)(II)$: $\cancel{ab} = r^2 \cancel{ab}(1-a)(1-b)$

$$(1-a)(1-b) = \frac{1}{r^2}$$

$$-(a+b) + ab = \frac{1}{r^2} - 1 \quad (**)$$

$$+2(a+b) - 2ab = 2 - \frac{2}{r^2}$$

$$(*) \quad -2(a+b) + 4ab = -1 - \frac{1}{r^2}$$

$+$

$$2ab = 1 - \frac{3}{r^2}$$

$$ab = \frac{1}{2} - \frac{3}{2r^2}$$

$(**)$: $a+b = ab - \frac{1}{r^2} + 1 = \frac{1}{2} - \frac{3}{2r^2} - \frac{1}{r^2} + 1$

$$a+b = \frac{3}{2} - \frac{5}{2r^2}$$

$(I) + (IV)$:

$$a+b = rb - rb^2 + ra - ra^2$$

$$(a+b) = r(a+b) - r\ (a^2+b^2)$$

$$r\ (a^2+b^2) = (r-1)(a+b)$$

$$a^2+b^2 = \left(\frac{r-1}{r}\right)(a+b)$$

$$(a+b)^2 = a^2+b^2+2ab$$

$$(a+b)^2 = (a+b)\left(\frac{r-1}{r}\right) + 2(ab)$$

$$(a+b)\left[(a+b) - \frac{r-1}{r}\right] - 2\ (ab) = 0$$

$$\left(\frac{3}{2} - \frac{5}{2r^2}\right)\left[\left(\frac{3}{2} - \frac{5}{2r^2}\right) - \frac{r-1}{r}\right] - 2\left(\frac{1}{2} - \frac{3}{2r^2}\right) = 0$$

$$\frac{3r^2-5}{2r^2}\left[\frac{3r^2-5-2(r-1)r}{2r^2}\right] - 1 + \frac{3}{r^2} = 0$$

$$3r^2 - 5 - 2r^2 + 2r = r^2+2r-5$$

$$\frac{(3r^2-5)(r^2+2r-5)}{4r^4} + \frac{3-r^2}{r^2}\frac{(4r^2)}{(4r^2)} = 0$$

$(3r^2-5)(r^2+2r-5)+12r^2-4r^4=0$ ④

$3r^4+6r^3-15r^2-5r^2-10r+25+12r^2-4r^4=0$

$-r^4+6r^3-8r^2-10r+25=0$

$-\underbrace{(r^2-4r+5)}\underbrace{(r^2-2r-5)}$

$=r^2-4r+4+1$

$=(r-2)^2+1$

always positive

$\dfrac{2\pm\sqrt{4+4.5}}{2} = \dfrac{2\pm\sqrt{24}}{2} = \dfrac{2\pm 2\sqrt{6}}{2} = 1\pm\sqrt{6}$

$1-\sqrt{6}<0$ $\qquad$ $1+\sqrt{6}\approx 3.449489742\cdots>0$

$(x^2+ax+b)(x^2+cx+d)$

$= x^4+ax^3+bx^2+cx^3+acx^2+bcx+dx^2+adx+bd$

$= x^4+\underbrace{(a+c)}_{=-6}x^3+\underbrace{(b+ac+d)}_{=8}x^2+\underbrace{(bc+ad)}_{=10}x+\underbrace{bd}_{=-25}$

$a+c=-6$ $\qquad$ $a+c=-6$

$(b+d)+ac=8 \Rightarrow ac=8$ $\qquad$ $a=-2,\ c=-4$

$bc+ad=10$

$bd=-25 \Rightarrow$ $\qquad$ $b=5,\ d=-5 \Rightarrow b+d=0$

$bc+ad=10$

$(5)(-4)+(-2)(-5) = -20+10=-10$ ✗

-2 $\qquad$ -4 ✓

$a=-4,\ c=-2$

$b=5 \qquad d=-5$